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ERRATA AU VOLUME V

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57^{9}	$(U\sigma_i\tau_i U^{-1})$	$(U\sigma_i\tau_i U^{-1}-\sigma_i\tau_i)$
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63^{8}	$\omega^2 nn'$	$\omega^2_{nn'}$
106_{17}	$d=\inf_{\delta>0,\{a_i,\ldots,a_p\}\subset\mathfrak{A}}\sup_{(p_1p_2)\in T}\lvert y(p_1)-y(p_2)\rvert$	$d=\inf_{\delta>0,\{a_1,a_2,\ldots a_k\}\subset\mathfrak{A},k=1,2,\ldots}\sup_{(p_1p_2)\in T_{\delta,a_1,a_2,\ldots,a_k}}\lvert y(p_1)-y(p_2)\rvert$
141_{5}	est un sous-groupe fermé	est un sous-groupe pur fermé
251_{6}	$\prod_{j\in A}$	$\prod_{j\in\bar{A}}$
270_{4}	$\bar{g}^2=\frac{4\pi}{c}C$	$\bar{g}^2=g^2\frac{4\pi}{3}C$
271^{11}	$\frac{526}{2^6\pi^{3/2}}\frac{\varkappa e\varkappa_N^2}{\mu}$	$\frac{526}{2^6\pi^{3/2}}\frac{\varkappa e\varkappa_N^2}{\mu^3}$
276^{12}	0 th	10 th
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502	Table I	
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XXII^{3}	$\prod_{j\in A}\sin m_j\,,dx_1,\ldots,dx_n$	$\prod_{j\in\bar{A}}\sin m_j x_j dx_1\ldots dx_n$

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MATHEMATICS

4-Quantifier Completeness: A Banach-Mazur Functional not Uniformly Partial Recursive

by

RICHARD M. FRIEDBERG

Presented by A. MOSTOWSKI on October 28, 1957

Let "Θ_u" denote a partial recursive function with Gödel number u, which takes finite sequences $\langle q_0, q_1, \ldots, q_n \rangle$ as arguments and natural numbers as values. Let "Θ_u^*" denote the functional which, on any recursive function f, takes the value $\Theta_u^* f = \lim_{n \to \infty} \Theta_u(\langle f(0), f(1), \ldots, f(n) \rangle)$, provided that Θ_u is defined on all finite sequences and that the limit exists for every recursive function.

Consider the predicate

Θ_u^* is defined on all recursive functions and is identical to some uniformly partial recursive functional. It may be written thus:

for every finite sequence, Θ_u is defined; and

there is a uniformly partial recursive functional Ψ, such that for each partial recursive function φ, there is an n for which either $\varphi(n)$ is undefined or for all $n' \geqslant n$, $\Theta_u(\langle \varphi(0), \ldots, \varphi(n') \rangle) = \Psi(\varphi)$.

Evidently this predicate can be written in the form $(Ex)(y)(Ez)(w)$ with recursive scope. We shall show that every predicate of that form is many-one reducible to it; i. e.,

THEOREM. *Given a recursive predicate on five variables, R, one can find effectively a recursive function r such that, for each u, $\Theta_{r(u)}^*$ is defined on all recursive functions, and $(Ex)(y)(Ez)(w) \sim R(u,x,y,z,w) \equiv [\Theta_{r(u)}^*$ is identical on recursive functions to a uniformly partial recursive functional$]$.*

Proof. Let $M(u,e,s,n,a) \equiv (x<e)(Ey<s)(z<n)(Ew<a)R(u,x,y,z,w)$. Then $(Ex)(y)(Ez)(w) \sim R(u,x,y,z,w) \equiv (Ee)(s)(En)(a) \sim M(u,e,s,n,a)$.

Given u, let $r(u)$ be the Gödel number of $\Theta_{r(u)}$, defined as follows: For any particular pair e,s, the definition of $\Theta_{r(u)}$ on all finite extensions of $\langle e,s \rangle$ is carried out in an infinite sequence of steps. A non-decreasing recursive function $t_{e,s}$ will be defined, signifying that when the con-

struction has reached any step a, $\Theta_{r(u)}$ has already been assigned the value 0 on all extensions of $\langle e,s,t\rangle$, where $t<t_{e,s}(a)$; $\Theta_{r(u)}$ is going to be assigned the value 0 on all extensions of $\langle e,s,t\rangle$, where $t>t_{e,s}(a)$, provided that $t_{e,s}(a')=t_{e,s}(a)$ for all $a'>a$; and we are presently building a recursive function $f_{e,s,t}$ with $t=t_{e,s}(a)$, which takes the values e,s,t on arguments $0,1,2$ respectively. e is interpreted as the Gödel number of a uniformly partial recursive functional Ψ_e. Our intention is that if s is the least number satisfying $(n)(Ea)\,M(u,e,s,n,a)$, then for some t we shall define $f_{e,s,t}$ on all arguments and contrive that $\Psi_e(f_{e,s,t})$ is either undefined or unequal to $\Theta^*_{r(u)}(f_{e,s,t})$.

At the outset, $\Theta_{r(u)}$ is assigned the value 0 on all sequences of length $\leqslant 3$. At each step a of the construction, one of Cases 0 to 5 is applied in furthering the definition of $\Theta_{r(u)}$ on extensions of $\langle e,s\rangle$.

Case 0: at step 0, start with $t_{e,s}(0)=0$, $f_{e,s,0}(0)=e, f_{e,s,0}(1)=s, f_{e,s,0}(2)=0$.

Case 1: at step $a>0$, if Case 2 has arisen at any previous step, then do nothing.

In the remaining Cases, we shall write "t" for "$t_{e,s}(a-1)$", "n" for *the least argument on which $f_{e,s,t}$ has not already been defined*, and "q" for *the least number q such that $\Theta_{r(u)}(f_{e,s,t}(0),\ldots,f_{e,s,t}(n-1),q\rangle)$ has not already been defined*".

Case 2: if, at step a, neither of the preceding Cases holds, and a value of $\Psi_e(f_{e,s,t})$ can be calculated in fewer than a steps on the basis of the values that $f_{e,s,t}$ takes on arguments $<n$, then let $f_{e,s,t}(n)=q$; $f_{e,s,t}(n')=0$, all $n'>n$; and give $\Theta_{r(u)}$ the value $\Psi_e(f_{e,s,t})+1$ on $\langle f_{e,s,t}(0),\ldots,f_{e,s,t}(n)\rangle$ and on all its extensions, and the value 0 on all other extensions of $\langle e,s\rangle$. Thus, $f_{e,s,t}$ is completely defined and $\Theta^*_{r(u)}(f_{e,s,t})\neq\Psi_e(f_{e,s,t})$.

Case 3: if, at step a, none of the preceding Cases holds, and $M(u,e,s-1,t,a)$, then give $\Theta_{r(u)}$ the value 0 on $\langle e,s,t\rangle$ and all its extensions, and let $t_{e,s}(a)=a$, $f_{e,s,a}(0)=e$, $f_{e,s,a}(1)=s$, $f_{e,s,a}(2)=a$. Thus, at the next step, we shall have a new value of t, and we shall start constructing a new function $f_{e,s,t}$.

Case 4: if, at step a, none of the preceding Cases holds, and $M(u,e,s,n,a)$, then give $\Theta_{r(u)}$ the value 0

on $\langle f_{e,s,t}(0),\ldots,f_{e,s,t}(n-1),q\rangle$ (but not on its extensions),

on $\langle e,s,a\rangle$ and all its extensions

and on $\langle f_{e,s,t}(0),\ldots,f_{e,s,t}(n-1),\ q'\rangle$ and all its extensions for every $q'\neq q$; also, let $f_{e,s,t}(n)=q$. Thus, at the next step, n will be increased by 1, and q will be 0.

Case 5: if, at step a none of the preceding Cases holds, then give $\Theta_{r(u)}$ the value 0

on $\langle e,s,a\rangle$ and all its extensions

and on $\langle f_{e,s,t}(0),\ldots,f_{e,s,t}(n-1),q\rangle$ and all its extensions. Thus, at the next step, q will be increased by 1.

The reader may easily verify that the foregoing construction, applied to each pair e,s, defines $\Theta_{r(u)}$ on all finite sequences, and that $\Theta^*_{r(u)}$ is defined on all functions. In fact, $\Theta^*_{r(u)}(f)=0$, except as determined otherwise by Case 2.

Now suppose that $(e)(Es)(n)(Ea)\ M(u,e,s,n,a)$. Then $\Theta^*_{r(u)}$ is not equivalent to any uniformly partial recursive functional. For, suppose Ψ_e is a uniformly partial recursive functional, with Gödel number e. Take the least s for which $(n)(Ea)\ M(u,e,s,n,a)$. Note that the definition of $f^{\exists\mathfrak{E}}_{e,s,t}$ progresses at each occurrence of Case 4 but starts all over with a new t at each occurrence of Case 3. For this particular e,s, we have $(Et)(a)\sim M(u,e,s-1,t,a)$ but $(n)(Ea)\ M(u,e,s,n,a)$. Therefore Case 3 occurs only a finite number of times, but Case 4 occurs an infinite number of times unless there is an occurrence of Case 2, which completes the definition of $f_{e,s,t}$ in one step. Hence, in any event, $f_{e,s,t}$ is completely defined for some t. But either $\Psi_e(f_{e,s,t})$ is not defined, or Case 2 eventually arises and consequently $\Theta^*_{r(u)}(f_{e,s,t})=\Psi_e(f_{e,s,t})+1$. Therefore $\Theta^*_{r(u)}$ is not equivalent on all recursive functions to Ψ_e. This holds for any e.

Suppose, on the other hand, that $(Ee)(s)(En)(a)\sim M(u,e,s,n,a)$. Let $e_0=(\mu e)(s)(En)(a)\sim M(u,e,s,n,a)$. For each $e<e_0$, let $s(e)=(\mu s)(n)(Ea)\ M(u,e,s,n,a)$. From the definition of M, it follows that $(s)(En)(a)\sim M(u,e,s,n,a)$ whenever $e\geqslant e_0$ and that $(n)(Ea)\ M(u,e,s,n,a)$ whenever $e<e_0$ and $s\geqslant s(e)$. (This is the reason for using M in place of R). If we know the value of e_0 and the values of $s(e)$ for all $e<e_0$, and if we know whether or not Case 2 arises in the construction for each $e<e_0$, $s=s(e)$ (all a finite quantity of information), we can describe a uniformly partial recursive functional Ψ which takes the same value as $\Theta^*_{r(u)}$ on every recursive function. Simply carry out the construction of $\Theta_{r(u)}$. Whenever $\Theta_{r(u)}$ is given a certain value on some finite sequence $\langle q_0,\ldots,q_{n-1}\rangle$ and on all its extensions, then give Ψ the same value on all functions φ such that $\varphi(x)=q_x$ for every $x<n$. In addition, if our previous information is that Case 2 does not occur in the construction for a particular pair $e<e_0$, $s=s(e)$, then give Ψ the value 0 on all functions φ such that $\varphi(0)=e$, $\varphi(1)=s$.

Obviously, Ψ takes the same value as $\Theta^*_{r(u)}$ on any recursive function on which Ψ is defined. We must show that Ψ is defined on every recursive function f. Consider the construction of $\Theta_{r(u)}$ for $e=f(0)$, $s=f(1)$. If Case 2 arises, $\Psi(f)$ surely becomes defined, since $\Theta_{r(u)}$ is defined at one step on all extensions of $\langle f(0),f(1)$. If Case 2 never arises, the pair $\langle f(0),f(1)\rangle$ falls into one of three categories.

A. $f(0)<e_0$, $f(1)=s(f(0))$. Then $\Psi(f)$ is set equal to 0 by the special clause in the definition of Ψ.

B. $f(0) < e_0$, $f(1) > s(f(0))$. Then $(t)(Ea)\, M(u, f(0), f(1) - 1, t, a)$. Hence, Case 3 occurs infinitely often, and eventually $\Psi(f)$ is defined either by Case 3 with $t = f(2)$ or by Case 4 or 5 with $a = f(2)$.

C. Otherwise. Then $(En)(a) \sim M(u, f(0), f(1), n, a)$. Therefore Case 3 occurs only a finite number of times, and after Case 3 has occurerd for the last time, Case 4 occurs only a finite number of times. Hence, eventually the construction settles down to an endless string of repetitions of Case 5. Note that before any step a (letting t, n, q be as defined above Case 2 in the construction), $\Psi(f)$ has been defined unless either $f(2) \geqslant a$ or $f(x) = f_{e,s,t}(x)$ for all $x < n$ and $f(n) \geqslant q$. The effect of an uninterrupted series of occurrences of Case 5 is that a and q grow boundlessly while t and n remain unchanged. Thus $\Psi(f)$ is eventually defined, whatever the values of f.

We have shown that $\Theta^*_{r(u)}$ is equivalent on recursive functions to some uniformly partial recursive functional if, and only if, $(Ee)(s)(En)(a) \sim M(u, e, s, n, a)$. Since the latter predicate is equivalent to $(Ex)(y)(Ez)(w) \sim R(u, x, y, z, w)$, the theorem is proven.

Corollary. *There is a Banach-Mazur functional which is not equivalent on all recursive functions to any uniformly partial recursive functional* *).

Proof. For any u, the statement, *Θ^*_u is a Banach-Mazur functional on recursive functions*, means: *Θ^*_u is defined on all recursive functions, and for each recursive two-argument function g there is a recursive function h such that (writing $f_m(n)$ for $g(m,n)$) $h(m) = \Theta^*_u(f_m)$ for every m.* This in turn can be written as the conjunction

For each finite sequence, Θ_u is defined; and

for each partial recursive φ, there is an n for which either $\varphi(n)$ is undefined or there is no $n' > n$ for which $\Theta_u(\langle \varphi(0), \ldots, \varphi(n') \rangle) \neq \Theta(\langle \varphi(0), \ldots, \varphi(n) \rangle)$; and

for each partial recursive two-argument function γ, there is either some m, n for which $\gamma(m,n)$ is undefined, or some partial recursive function η such that for each m, n, $\eta(m)$ is defined and is equal to $\Theta_u(\langle \gamma(m,0), \ldots, \gamma(m,n') \rangle)$ for some $n' > n$.

The last *some* has the force of *all sufficiently high* because of the second member of the conjunction. The conjunction may be put in the form $(x)(Ey)(z)(Ew)\, R(u, x, y, z, w)$ with recursive R. For this R, obtain r from the preceding theorem. Apply the recursion theorem to r to obtain a number u for which Θ_u is identical to $\Theta_{r(u)}$. Then the preceding theorem says that

[Θ^*_u is defined on all recursive functions and is equivalent on them to some uniformly partial recursive functional] $\not\equiv$

*) This answers the question in footnote 4 of [1].

$\not\equiv$ [Θ_u^* *is defined on all recursive functions and is a Banach-Mazur functional*].

The left side of this inequivalence implies the right. Hence the left is false and the right is true.

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MATHEMATICS

On the Potency of Compact Spaces and the First Axiom of Countability

by

S. MRÓWKA

Presented by K. KURATOWSKI on October 25, 1957

Let us note that there exists a *countable* space which does not satisfy the first axiom of countability at any of its points. For instance, let X be the space of all polynomials with rational coefficients; the topology in X is the Tychonoff topology, i. e. sets of the form

$$[f; x_1, \ldots, x_k; \varepsilon] = \{g \in X: |f(x_i) - g(x_i)| < \varepsilon; \quad i = 1, \ldots, k\},$$

where $x_1, \ldots x_k$ are any points of the real line, and $\varepsilon > 0$ forms a basis of neighbourhoods of an element $f \in X$. Clearly, X is countable. On the other hand, let

$$U_n = [f; x_1^{(n)}, \ldots, x_{k_n}^{(n)}; \varepsilon_n] \qquad n = 1, 2, \ldots$$

be any sequence of neighbourhoods of an element $f \in X$. There is a point x with $x \neq x_i^{(n)}$ for $n = 1, 2, \ldots; \ i = 1, \ldots, k_n;$ let $U = [f; x, 1]$. Then U is a neighbourhood of f and $U_n \not\subset U$ for $n = 1, 2, \ldots$ Thus, f does not satisfy the first axiom of countability.

Alexandroff and Urysohn have proved [1] that if X is a compact space and $p \in X$, then p satisfies the first axiom of countability if, and only if, $\{p\}$ is a G_δ-set. It follows that each compact space having no point satisfying the first axiom of countability is uncountable. Some generalizations of this fact are given below.

THEOREM. *Let X be any compact space which has no point satisfying the first axiom of countability. Then $\bar{X} \geqslant 2^{\aleph_1}$.*

Proof. We begin with the following statement: if F is a closed G_δ set lying in an arbitrary, completely regular space and having at least two points, then there are two disjoint non-empty closed G_δ-sets contained in F. In fact, there is a continuous real-valued function f defined on F, such that $0, 1 \in f(F)$. Let $F_0 = f^{-1}(0)$, $F_1 = f^{-1}(1)$. Then the sets F_1, F_2

are G_δ's with respect to F; therefore they are G_δ's with respect to the whole space.

Let us denote by $\mathfrak{F}$ the family of all closed non-empty G_δ — subsets of X. Since no point of X satisfies the first axiom of countability, each member of $\mathfrak{F}$ contains at least two points. Therefore, in virtue of the preceding remark and of the axiom of choice, there are two functions $\mathfrak{f}_0$ and $\mathfrak{f}_1$ on $\mathfrak{F}$ into itself, such that $\mathfrak{f}_0(F) \cup \mathfrak{f}_1(F) \subset F$ and $\mathfrak{f}_0(F) \cap \mathfrak{f}_1(F) = 0$ for each F in $\mathfrak{F}$.

Let $A = \{\alpha_0, \alpha_1, \ldots, \alpha_\xi, \ldots\}_{\xi<\Omega}$ be any transfinite sequence of 0's and 1's. We define a decreasing transfinite sequence $\{F_0^{(A)}, F_1^{(A)}, \ldots, F_\xi^{(A)}, \ldots\}_{\xi<\Omega}$ of members of F in the following way:

1° $F_0^{(A)} = \mathfrak{f}_{\alpha_0}(X)$;

2° suppose that, for some $\xi < \Omega$, the sets $F_0^{(A)}, \ldots, F_\eta^{(A)}, \ldots$ are defined for each $\eta < \xi$ in such a way that $F_\eta^{(A)} \in F$ and $F_0^{(A)} \supset \ldots \supset F_\eta^{(A)} \supset \ldots$

Let

$$F = \bigcap_{\eta<\xi} F_\eta^{(A)}.$$

Clearly, F is non-empty and F, being the intersection of a countable infinity of members of $\mathfrak{F}$, is a member of $\mathfrak{F}$. Let $F_\xi^{(A)} \quad \mathfrak{f}_{\alpha_\xi}(F)$.

It is plain from the above definition that if

$$B = \{\beta_0, \beta_1, \ldots, \beta_\xi, \ldots\}_{\xi<\Omega}$$

and $\beta_\xi \neq \alpha_\xi$, then $F_\xi^{(A)} \cap F_\xi^{(B)} = 0$.

Let

$$F^{(A)} = \bigcap_{\xi<\Omega} F_\xi^{(A)}.$$

Then $F^{(A)} \neq 0$ and $F^{(A)} \cap F^{(B)} = 0$ for $A \neq B$. Since the set of all transfinite sequences (of the type Ω) of 0's and 1's is of the power $2^{\aleph_1}$, $\overline{X} \geqslant 2^{\aleph_1}$. Thus, the theorem is proved.

Note that, if F is any closed G_δ-subset of X, then each point $p \in F$ which satisfies the first axiom of countability with respect to F also satisfies the axiom with respect to X. Thus, we obtain:

COROLLARY 1. *If F is any closed non-empty G_δ-subset of a compact space and F contains no point satisfying the first axiom of countability, then $\overline{F} \geqslant 2^{\aleph_1}$.*

In particular:

COROLLARY 2. *Each closed non-empty G_δ-set contained in $\beta X - X$, where X is any completely regular space and βX denotes the maximal Stone-Čech compactification of X, is of the power $\geqslant 2^{\aleph_1}$.*

It is actually known that no point of $\beta X - X$ satisfies the first axiom of countability.

If X is any uncountable space, and $X_0 = \{p_1, p_2, \ldots\}$ is any countable subset of X, then there is a non-empty closed G_δ-subset of X which is disjoint from X_0. In fact, let p_0 be any point of X which does not belong to X_0, and let f_n be a continuous function on X into the unit interval $[0, 1]$ such that $f_n(p_0) = 0$, $f_n(p_n) = 1$. Setting $f(p) = \sum \frac{1}{2^n} \cdot f_n(p)$, we have $f(p_0) = 0$ and $f(p) \neq 0$ for p in X_0, and the function f is continuous; therefore $F = f^{-1}(0)$ is a closed non-empty G_δ-set disjoint from X_0. It follows that each uncountable space, having at most countably many points satisfying the first axiom of countability, contains a closed non-empty G_δ-set without points satisfying the axiom. Therefore, by Corollary 1, we obtain:

COROLLARY 3. *Each uncountable compact space X with $X < 2^{\aleph_1}$ contains an uncountable infinity of points satisfying the first axiom of countability.*

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MATHEMATICS

On Contact Points of Two Integrals of a Partial Differential Equation of the First Order

by

A. PLIŚ

Presented by T. WAŻEWSKI on October 25, 1957

In this note we shall deal with the partial differential equation

$$z_x = f(x, y_1, \ldots, y_n, z, z_{y_1}, \ldots, z_{y_n}) . \tag{1}$$

It is well known that, for f of class C^2 and for integrals of (1) of class C^2, the following property is satisfied:

PROPERTY **P**: *If two integrals z, s are tangent at a point, then they are tangent along the corresponding characteristic.*

Let us generalize that theorem. We shall prove property **P** for $n = 1$ by assuming that the functions z, s, f are of class C^1 and that f satisfies an assumption (assumption B) which is weaker than that of class C^2 (Theorem 1). For $n > 1$ we assume that f is of class C^2 and z, s satisfy (7) (Theorem 2).

Moreover, we shall give an example (example E) showing that property **P** may not be satisfied for f of class C^1. In example E there exist two polynomial integrals of the equation

$$z_x = f(x, y, z, z_y) , \tag{2}$$

tangent at one and only one point, and f is a (rational) function of class C^1.

Assumption A. Let the integrals $z(x, y)$, $s(x, y)$ of (2) be of class C^1 in a neighbourhood of point $(0, 0)$ and function $f(x, y, z, q)$ be of class C^1 in a neighbourhood of point $x = 0$, $y = 0$, $z = c = z(0, 0)$, $q = k = z_y(0, 0)$ *).

*) The existence of derivative f_x is not necessary.

Assumption B. Let the solution $y(x), z(x), q(x)$ of the characteristic equations

$$\begin{aligned} y' &= -f_q(x, y, z, q), \\ z' &= f(x, y, z, q) - qf_q(x, y, z, q), \\ q' &= f_y(x, y, z, q) + qf_z(x, y, z, q) \end{aligned} \tag{3}$$

by the initial conditions

$$y(0) = 0, z(0) = c, q(0) = k$$

be unique.

Theorem 1. *Let the assumptions* A, B *be satisfied. If the conditions*

$$s(x, y(x)) = z(x, y(x)), \quad s_y(x, y(x)) = z_y(x, y(x)) \tag{4}$$

are satisfied for $x = 0$, *then they are satisfied in a neighbourhood of point* $x = 0$.

Proof. Theorem 1 is a corollary of the following Theorem T proved in [1]:

Theorem T. *If assumption* A *is satisfied, there exists a solution* $\eta(x), \zeta(x), \gamma(x)$ *of* (3) *which fulfils the condition* $\eta(0) = 0$ *and the identities*

$$\zeta(x) = z(x, \eta(x)), \quad \gamma(x) = z_y(x, \eta(x)) \tag{5}$$

in a neighbourhood N *of point* $x = 0$.

In view of assumption B, the identities

$$y(x) = \eta(x), \quad z(x) = \zeta(x), \quad q(x) = \gamma(x)$$

hold in N. Hence, the identities

$$z(x, y(x)) = z(x), \quad z_y(x, y(x)) = q(x) \tag{6}$$

hold in N. The integral $z = s(x, y)$ satisfies also relations (6), and Theorem 1 follows.

In a similar manner we can prove the following theorem, if we apply Remarks 2,3 contained in paper [2]:

Theorem 2. *Let the integrals* $z(x, y_1, \ldots, y_n)$, $s(x, y_1, \ldots, y_n)$ *of* (1) *be of class* C^1 *and satisfy the inequalities*

$$|z_{y_j}(x, y_1, \ldots, y_n) - z_{y_j}(x, y_1^*, \ldots, y_n^*)| \leqslant K \sum_{i=1}^{n} |y_i - y_i^*|^s, \tag{7}$$

$(j = 1, \ldots, n;\ K > 0;\ s \geqslant \frac{1}{2})$ *in a neighbourhood of point* $x = 0, y_i = 0$ $(i = 1, \ldots, n)$. *Let the function* $f(x, y_1, \ldots, y_n, z, q_1, \ldots, q_n)$ *be of class* C^2

in a neighbourhood of point $B: x = 0, y_i = 0\ (i = 1, \ldots, n), z = z(0, 0, \ldots, 0)$, $q_i = z_{y_i}(0, 0, \ldots, 0)\ (i = 1, \ldots, n)$. *Let us denote by* $y_1(x), \ldots, y_n(x)$, $z(x)$, $q_1(x), \ldots, q_n(x)$ *the solution of the characteristic differential equations:*

$$y_i' = -f_{q_i}(x, y_1, \ldots, y_n, z, q_1, \ldots, q_n), \qquad (i = 1, \ldots, n)$$

$$z' = f(\ldots) - \sum_{k=1}^{n} q_k f_{q_k}(\ldots),$$

$$q_i' = f_{y_i}(\ldots) + q_i f_z(\ldots), \qquad (i = 1, \ldots, n)$$

passing through point B.

Under these assumptions, if the conditions

$$s(x, y_1(x), \ldots, y_n(x)) = z(x, y_1(x), \ldots, y_n(x)),$$

$$s_{y_j}(x, y_1(x), \ldots, y_n(x)) = z_{y_j}(x, y_1(x), \ldots, y_n(x)),$$

$(j = 1, \ldots, n)$ *are satisfied for* $x = 0$, *then they are satisfied in a neighbourhood of point* $x = 0$.

EXAMPLE E *). We take

$$g(x, y, q) = 4x^3 y A(q(x^4 + 3y^2)^{-1}) \quad \text{for} \quad (x, y) \neq (0, 0),$$
$$g(0, 0, q) = 0,$$

where $A(s)$ is a function of class C^∞ bounded together with $|A'(s).s|$ on the interval $(-\infty, +\infty)$ and such that $A(0) = 0$, $A(1) = 1$. It is evident that the functions $z^*(x, y) = 0$, $s^*(x, y) = x^4 y + y^3$ satisfy (2) for $f = g$, and the surfaces $z = z^*(x, y)$, $z = s^*(x, y)$ are tangent at the point $x = 0$, $y = 0$, $z = 0$, and are not tangent at any other point. It is easy to verify that function $g(x, y, q)$ is of class C^1 in the whole space, and of class C^∞ for $(x, y) \neq (0, 0)$; for $A(s) = 2 - 2(1 + s^2)^{-1}$ it is a rational function. Thus equation $z_x = g(x, y, z_y)$ has the required properties.

Remark 1. For $A(s)$ satisfying additionally the identity $A(s) = s$ in the interval $\langle -1, 1 \rangle$ there exists the one-parameter family of integrals of (2) $(f = g)$, $z(x, y; s) = (x^4 y + y^3) \cdot s$, $|s| < 1$, tangent to one another at one point only.

Remark 2. Let the function $A(s)$ from example E satisfy the additional condition $A'(0) = A'(1) = 0$. It is easy to verify that $y = 0$, $z = 0$, $q = 0$ is the unique solution of (3) $(f = g)$ satisfying condition $y(0) = 0$ and identities (5) for $z = z^*(x, y)$, and that $y = 0$, $z = 0$, $q = x^4$ is the unique solution of (3) satisfying the condition $y(0) = 0$ and identities (5) for $z = s^*(x, y)$. Thus, there does not exist a solution $\eta(x), \zeta(x)$,

*) This example was communicated on June 18. 1957 at the meeting of the Cracow Section of the Polish Mathematical Society.

$\gamma(x)$ of (3) satisfying the condition $\eta(0) = 0$ and identities (5) for both the functions $z = z^*(x, y)$ and $z = s^*(x, y)$ simultaneously.

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MATHÉMATIQUE

Une remarque à propos de deux notes de Z. Szmydt

par

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Présenté par T. WAŻEWSKI le 28 Octobre 1957

Dans les deux notes ([2] et [3]) sur les équations différentielles partielles hyperboliques, Mlle Szmydt a utilisé un lemme ([2], p. 572) bien simple et maniable sur l'existence d'un point fixe d'une certaine transformation définie dans un produit cartésien d'espaces métriques. Nous allons donner une démonstration plus rapide dudit lemme, ce qui va conduire à sa généralisation.

LEMME. *Admettons, pour* $i = 1, 2$, *que* E_i *et* $\delta_i(\xi_i, \eta_i)$ *désignent un espace métrique complet et la distance de deux éléments de* E_i *et supposons que*

$$\widetilde{\xi}_i = \Phi_i(\xi_1, \xi_2)$$

soit une transformation du produit cartésien $E^* = E_1 \times E_2$ *dans* E_i *telle que*

$$\delta_i(\widetilde{\xi}_i, \widetilde{\eta}_i) \leqslant a_{i1} \cdot \delta_1(\xi_1, \eta_1) + a_{i2} \cdot \delta_2(\xi_2, \eta_2), \tag{1}$$

pour tout couple de points $(\xi_j, \eta_j) \in E_j \times E_j$, $j = 1, 2$, *où* a_{ij} *sont des nombres positifs.*

Alors, si

$$a_{11} < 1, \quad a_{22} < 1 \quad \text{et} \quad a_{12}a_{21} < (1 - a_{11})(1 - a_{22}), \tag{2}$$

il existe un seul point fixe de la transformation

$$(\widetilde{\xi}_1, \widetilde{\xi}_2) = (\Phi_1(\xi_1, \xi_2), \Phi_2(\xi_1, \xi_2)) \tag{3}$$

de l'espace E^* *dans* E^*.

En effet, en tenant compte de (2), il existe trois nombres positifs $\mathcal{L}$, α et β tels que

$$a_{11} \leqslant \mathcal{L} < 1, \; a_{22} \leqslant \mathcal{L} \quad \text{et} \quad \frac{a_{12}}{\mathcal{L} - a_{22}} \leqslant \frac{\beta}{\alpha} \leqslant \frac{\mathcal{L} - a_{11}}{a_{21}}. \tag{4}$$

En introduisant la distance

$$\delta^*[(\xi_1, \xi_2), (\eta_1, \eta_2)] = \alpha \cdot \delta_1(\xi_1, \eta_1) + \beta \cdot \delta_2(\xi_2, \eta_2), \tag{5}$$

E^* devient évidemment un espace métrique complet et on obtient, d'après (1), (4) et (5), la relation simple

$$\delta^*[(\tilde{\xi}_1, \tilde{\xi}_2), (\tilde{\eta}_1, \tilde{\eta}_2)] \leqslant \mathcal{L} \cdot \delta^*[(\xi_1, \xi_2)(\eta_1, \eta_2)],$$

où $0 < \mathcal{L} < 1$. Donc, en vertu du théorème bien connu de Banach, il existe un seul point fixe de la transformation (3):

$$(\overset{\circ}{\xi}_1, \overset{\circ}{\xi}_2) = (\Phi_1(\overset{\circ}{\xi}_1, \overset{\circ}{\xi}_2), \Phi_2(\overset{\circ}{\xi}_1, \overset{\circ}{\xi}_2)) \in E^*. \tag{6}$$

En particulier, le lemme cité ci-dessus s'obtient pour

$$a_{11} = a_{22} - c > 0, \quad a_{12} = c + A, \quad a_{21} = c - B, \quad A \geqslant 0, \quad B \geqslant 0$$

et

$$\Phi_1(\xi_1, \xi_2) = \Phi(\xi_1, \xi_2, \xi_2), \Phi_2(\xi_1, \xi_2) = \Psi(\xi_1, \xi_2, \xi_1),$$

mais les inégalités (2) se réduisent alors à la condition

$$c < \frac{1 - AB}{A + B + 2} \tag{7}$$

un peu plus faible que la condition

$$c < \frac{1 - AB}{2 + A + (1 + A)\dfrac{B}{1 - AB}}$$

contenue implicitement dans la démonstration esquissée par Mlle Szmydt ([2], p. 573).

Observons encore que les conditions (2) qui figurent dans notre énoncé du lemme sont les plus faibles possibles. En effet, supposons que $E_1 = E_2 = [0, \infty)$ soit l'ensemble de tous les nombres réels non négatifs et que

$$\Phi_1(\xi_1, \xi_2) = a_{11}\xi_1 + a_{12}\xi_2 + 1 \quad \text{et} \quad \Phi_2(\xi_1, \xi_2) = a_{21}\xi_1 + a_{22}\xi_2.$$

On vérifie sans peine que le système d'équations (6) n'a pas de solutions dans $E_1 \times E_2$ lorsque $a_{11} \geqslant 1$ ou bien lorsque $a_{11} < 1$, $a_{22} < 1$ et $a_{12}a_{21} \geqslant (1 - a_{11})(1 - a_{22})$; le cas, où $a_{22} \geqslant 1$, est tout à fait analogue. Ainsi, l'inégalité (7) ne peut être affaiblie.

En appliquant dans les démonstrations des théorèmes ([2], p. 572, [3], p. 579, et aussi [1], p. 581) la modification du lemme de Mlle Szmydt que nous venons de démontrer, on obtient un léger affaiblissement des conditions auxquelles sont assujetties les constantes figurant dans ces théorèmes.

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MATHEMATICS

On the Approximate Solutions of Non-Linear Functional Equations in Banach Spaces

by

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Presented by W. ORLICZ on November 3, 1957

Papers [1] and [2] contain an iterative method for solving non-linear functional equations in Banach spaces. This method may also be regarded as an extension of Newton's well-known classical method. But this extension is essentially different from the method given by L. V. Kantorovitch [3], the latter being applicable only to operator equations in Banach spaces.

The present note concerns the examination of the method defined in papers [1] and [2]. A fundamental role is here played by the majorant method, which was applied by Kantorovitch [3] to an examination of Newton's method. The application of the majorant method to Newton's iterative process by Kantorovitch [3] consists in the application of the majorant method to a certain ordinary successive approximation process. But this is not possible in our case. Hence, we must consider some modification of Kantorovitch's method.

The application of the majorant method makes it possible to obtain new conditions for the convergence of the process defined in [2]. These conditions guarantee also the existence of a solution of the equation here considered.

Moreover, we obtain in this way a generalization of some results previously proved and a simplification of the theory.

Let X be a Banach space and let $F(x)$, $x \in X$, be a non-linear continuous functional defined on the closed solid sphere $S(x_0, r)$ of X with centre x_0 and radius r.

Consider the non-linear functional equation

$$F(x) = 0 . \tag{1}$$

Let us assume that $F(x)$ is differentiable in Fréchet's sense in the sphere $S(x_0, r)$, and denote by $f(x) = F'(x)$ the Fréchet differential of $F(x)$.

Putting $f_n = f(x_n) = F'(x_n)$ we choose a sequence of elements $y_n \in X$ such that

$$\|y_n\| = 1, \quad f_n(y_n) = \|f_n\| \quad n = 0, 1, 2, \ldots \tag{2}$$

provided that such a choice is possible.

The iterative process for solving Eq. (1) is defined as in papers [1] and [2]:

$$x_1 = x_0 - \frac{F(x_0)}{f_0(y_0)} y_0, \quad x_{n+1} = x_n - \frac{F(x_n)}{f_n(y_n)} y_n. \tag{3}$$

Let us further assume that there exists the second Fréchet differential $F''(x)$ of $F(x)$ in the sphere $S(x_0, r)$.

Consider now the real equation

$$Q(x) = 0, \tag{4}$$

where $Q(z)$ is a real function of the real variable z, being twice continuously differentiable in (z_0, z').

Following Kantorovitch [3] let us say that Eq. (1) possesses a real majorant equation (4), if the following conditions are satisfied:

1° $Q'(z_0) \neq 0$ and $B_0 = -\dfrac{1}{Q'(z_0)} > 0$;

2° $|F(x_0)| \leqslant Q(z_0)$;

3° $\dfrac{1}{\|F'(x_0)\|} \leqslant B_0$;

4° $\|F''(x)\| \leqslant Q''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0$.

The following theorem gives conditions of the convergence of process (3) which also suffice for the existence of a solution of Eq. (1).

THEOREM. *If Eq.* (1) *possesses a real majorant equation* (4), *and if Eq.* (4) *has a real root in the segment* (z_0, z'), *then Eq.* (1) *has a solution* x^* *in the sphere* $S(x_0, r)$, *where* $r = z' - z_0$, *and the sequence of approximate solutions* x_n *constructed by process* (3) *converges to it.*

Proof. Let us define the sequence of approximate solutions z_n of Eq. (4) by Newton's classical process, i. e.

$$z_{n+1} = z_n - \frac{Q(z_n)}{Q'(z_n)}, \quad n = 0, 1, 2, \ldots \tag{5}$$

or

$$z_{n+1} = V_n(z_n), \tag{6}$$

where $V_n(z) = z - \dfrac{Q(z)}{Q'(z_n)}$.

Equation $V_0(z) = z$ is equivalent to Eq. (4).

Since

$$\int_{z_0}^{z} BQ''(z)\,dz = BQ'(z) - BQ'(z_0) = 1 + BQ'(z) \qquad V'(z),$$

we have, by 4°,

(7) $$V_0'(z) \geqslant 0\,.$$

Hence, if z^* is a root of Eq. (4) and z^* lies in (z_0, z'), we have, by (7),

(8) $$z_1 = V_0(z_0) \leqslant V_0(z^*) = z^* \leqslant z'\,.$$

Since, by (3), $\|x_1 - x_0\| = \dfrac{|F(x_0)|}{f_0}$, it follows from 1°, 2°, 3° and Eq. (5) that

(9) $$\|x_1 - x_0\| \leqslant z_1 - z_0\,.$$

We shall now show that all conditions 1°-4° are satisfied if element x_0 is replaced by x_1.

Using the analogue of Taylor's formula in the integral form we obtain

$$F(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) + \int_{x_0}^{x_1} F''(\bar{x})(x_1 - \bar{x})\,d\bar{x}\,.$$

Since, by (3), $F(x_0) + F'(x_0)(x_1 - x_0) = 0$, we have

(10) $$F(x_1) = \int_{x_0}^{x_1} F''(x)(x_1 - \bar{x})\,d\bar{x}\,.$$

Analogously we obtain

(11) $$Q(z_1) = \int_{z_0}^{z_1} Q''(\bar{z})(z_1 - \bar{z})\,d\bar{z}\,.$$

Since integral (10) is majorised by integral (11), we have

(12) $$|F(x_1)| \leqslant Q(z_1)$$

and condition 2° is satisfied for x_1.

Since $Q''(z) \geqslant 0$, the derivative $Q'(z)$ increases still preserving the minus sign at point z_1. Hence, condition 1° is also satisfied for x_1.

Further, we have

$$\|F'(x)\| \geqslant \|F'(x_0)\| - \|F'(x_1) - F'(x_0)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\|F'(x_1) - F'(x_0)\|}{\|F'(x_0)\|}\right).$$

Using the abstract analogue of the fundamental formula of the integral calculus, we obtain by 4° and Eq. (9)

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\left\|\int_{x_0}^{x_1} F''(x)\,dx\right\|}{\|F'(x_0)\|}\right) \geqslant \|F'(x_0)\| \left(1 - \frac{\int_{z_0}^{z_1} Q''(z)\,dz}{|F'(x_0)|}\right) =$$

$$\geqslant \|F'(x_0)\| \left(1 - \frac{Q'(z_1) - Q'(z_0)}{Q'(z_0)}\right) = \|F'(x_0)\| \frac{Q'(z_1)}{Q'(z_0)}\,.$$

Thus, we obtain:

$$\frac{1}{\|F'(x_1)\|} \leqslant -\frac{1}{Q'(z_1)}. \tag{13}$$

Finally, it is easy to see that condition 4° is also satisfied for x_1. In fact, if

$$\|x-x_1\| \leqslant z-z_1 \leqslant z'-z_1,$$

then we have, by (9),

$$\|x-x_0\| \leqslant \|x-x_1\| \leqslant \|x_1-x_0\| \leqslant z-z_1+z_1-z_0 = z-z_0 \leqslant z'-z_0.$$

In the same way as above we infer that

$$\|x_2-x_1\| \leqslant z_2-z_1 \tag{14}$$

and

$$z_2 \leqslant z'. \tag{15}$$

By induction we obtain

$$\|x_{n+p}-x_n\| \leqslant z_{n+p}-z_n \tag{16}$$

and

$$z_n \leqslant z^* \leqslant z',$$

where z^* is the nearest to z_0 root of Eq. (4).

Since the sequence of z_n is increasing and bounded, it converges to solution z^* of Eq. (4).

It follows immediately from (16) that the sequence of the approximate solutions x_n is convergent to a certain element x^*, being a solution of Eq. (1), by (3).

Remark 1. It follows from (16) that

$$\|x_n-x^*\| \leqslant z^*-z_n. \tag{17}$$

Hence, the error estimate is given by formula (17).

Remark 2. Condition 2° can be replaced by condition

$$\|x_1-x_0\| \leqslant z_1-z_0.$$

COROLLARY 1. *Let us assume that the following conditions are satisfied:*

1° $\dfrac{1}{\|F'(x_0)\|} \leqslant B,$

2° $|F(x_0)| \leqslant \dfrac{\eta}{B},$

3° $\|F''(x)\| \leqslant \dfrac{K(r)}{B}$ *if* $\|x-x_0\| \leqslant r.$

If z^* is a positive root of the equation

$$Q(z) = \int_0^z d\varrho \int_0^\varrho K(r)\,dr - z + \eta = 0,$$

then Eq. (1) *has a solution* x^* *in the sphere* $S(x_0, z^*)$, *and the sequence of approximate solutions* x_n *constructed by process* (3) *converges to it. For the error estimate we have the inequality* (17).

By way of proof it is sufficient to observe that Eq. (1) possesses the majorant equation $\frac{1}{B}Q(z) = 0$, since $-Q'(z_0) = 1$, $Q(z_0) = \eta$ and $Q''(z) = K(z)$, provided $z_0 = 0$.

COROLLARY 2. *Let us assume that the following conditions are satisfied:*

1° *for the Fréchet differential* $F'(x_0)$, *at the initial approximate solution* x_0, *the inequality*

$$\frac{1}{\|F'(x_0)\|} \leqslant B$$

is fulfilled;

2° *the value* $F(x_0)$ *satisfies the inequality*

$$F(x_0) \leqslant \eta;$$

3° *there exists a constant* K, *such that the norm of the second Fréchet differential is bounded by it in the sphere* $S(x_0, r)$, *i. e.*

$$\|F''(x)\| \leqslant K \quad \text{for} \quad x \in S(x_0, r);$$

4° *the constants* B, η *and* K *are subject to condition*

$$h = B^2\eta K \leqslant \tfrac{1}{2};$$

and

$$r = \frac{1 - \sqrt{1-2h}}{h} B\eta.$$

Then Eq. (1) *has a solution* x^* *belonging to sphere* $S(x_0, r)$, *and the sequence of approximate solutions* x_n *defined by process* (3) *converges to it. For the error estimate we have the formula*

$$\|x_n - x^*\| \leqslant \frac{1}{2^{n-1}}(2h)^{2^n-1} B\eta.$$

The proof [2] results from the fact that Eq. (1) possesses the majorant equation

$$\frac{1}{2}Kz^2 - \frac{1}{B}z + \eta = 0,$$

for which condition 4° assures the existence of real roots. Putting here $z_0 = 0$ we see that all conditions of the above Theorem are satisfied.

Remark 3. A more accurate error estimate for the first approximate solution x_1 can be given by the formula

$$\|x^* - x_1\| \leqslant \frac{1 - \sqrt{1-2h_1}}{h_1} B_1\eta_1, \tag{18}$$

where

$$|F(x_1)| \leqslant \eta_1; \quad B_1 = \frac{B}{1-h} \quad \text{and} \quad h_1 = B_1^2 K \eta_1 .$$

In fact, we have by (13)

$$\frac{1}{\|F'(x_1)\|} \leqslant -\frac{1}{Q'(z_1)} = -\frac{1}{Kz_1 - \frac{1}{B}} = \frac{-B}{B^2 K\eta - 1} = \frac{B}{1-h} = B_1 .$$

But condition 3° of Corollary 2 remains the same, hence the assertion results from Corollary 2.

Formula (18) can also be written in the form

$$\|x^* - x_1\| \leqslant \frac{(1-h) - \sqrt{(1-h)^2 - 2B^2 K\eta_1}}{BK} .$$

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МАТЕМАТИКА

А. ГРАНАС

О НЕПРЕРЫВНЫХ ОТОБРАЖЕНИЯХ ОТКРЫТЫХ МНОЖЕСТВ В БАНАХОВЫХ ПРОСТРАНСТВАХ

Представлено К. БОРСУКОМ 4 ноября 1957

В настоящей заметке рассматриваются непрерывные отображения открытых множеств, расположенных в банаховом пространстве. Из основного доказываемого здесь результата (Теорема 1) вытекает, в частности, известная теорема об инвариантности области, а также теоремы являющиеся обобщением на бесконечномерный случай известных результатов К. Борсука, касающихся непрерывных отображений евклидовых пространств. Идея доказательства состоит в применении теоремы К. Борсука об антиподах.

Определения и предварительные замечания

1. Пусть X — банахово пространство. Если $x_0 \in X$, то положим $V_\infty(x_0;\varrho) = \underset{x\in X}{E}(\|x-x_0\| < \varrho)$, $S_\infty(x_0,\varrho) = \underset{x\in X}{E}(\|x-x_0\| = \varrho)$ ($V_\infty(x_0,\varrho)$ есть открытый шар радиуса ϱ с центром x_0, а $S_\infty(x_0,\varrho)$ — его граница). Положим далее $V_\infty^0 = V_\infty(0,1)$, $S_\infty^0 = S_\infty(0,1)$.

Непрерывное отображение $f: X_1 \to X_2$, где $X_1, X_2 \subset X$ назовем *ε-отображением* (ε — вещественное положительное число), если выполнено следующее условие:

$$(\alpha) \qquad [f(x_1)=f(x_2)] \to [\|x_1-x_2\|<\varepsilon] \quad \text{для любых } x_1, x_2 \in X.$$

Отображение $f: X_1 \to X_2$ назовем *ε-отображением в узком смысле*, если для некоторой постоянной $\eta>0$ выполнено условие:

$$(\beta) \qquad [\|f(x_1)-f(x_2)\|<\eta] \to [\|x_1-x_2\|<\varepsilon] \quad \text{для любых } x_1, x_2 \in X.$$

Пусть O — открытое множество расположенное в X. Для непрерывного отображения $f: O \to X$ рассмотрим следующее свойство:

(α^*) *для любой точки $x \in O$ существует такое положительное число ε_x, что замыкание $\overline{V_\infty(x,\varepsilon_x)}$ шара $V_\infty(x,\varepsilon_x)$ содержится в O и f рассматриваемое на $\overline{V_\infty(x,\varepsilon_x)}$ есть ε_x — отображение.*

2. В дальнейшем нам будет удобно пользоваться терминологией теории векторных полей. Непрерывную функцию $f: X_0 \to X$, где X_0 есть какое либо подмножество X, можно рассматривать как векторное поле в X, заданное на X_0. Непрерывное отображение $f: X_0 \to X$ представимое в виде

$$f(x) = x - F(x), \tag{1}$$

где $F: X_0 \to X$ есть вполне непрерывный оператор, будем называть вполне непрерывным векторным полем на X_0. В дальнейшем мы будем рассматривать исключительно отображения вида (1).

Если вполне непрерывное векторное поле $f: \overline{V}_\infty \to X$, $V_\infty = V_\infty(x_0, \varrho)$ не исчезает на $S_\infty = S_\infty(x_0, \varrho)$, т. е. $f(x) \neq 0$ для $x \in S_\infty$, то через $\chi(f, S_\infty)$ будем обозначать *характеристику* поля f на сфере S_∞ (см. [7] стр. 187).

Если $f: \overline{V}_\infty \to X$ произвольное отображение вида (1), причем $f(x_0) \in f(S_\infty)$, то определяемая по Лере-Шаудеру [6] *степень* $d(f, V_\infty, f(x_0))$ *отображения* f *в точке* $f(x_0)$ равна характеристике $\chi(\varphi, S_\infty)$ вполне непрерывного векторного поля φ определяемого по формуле: $\varphi(x) = f(x) - f(x_0)$.

Пусть $\varphi_0, \varphi_1: S_\infty \to X$ два вполне непрерывных векторных поля на S_∞ без нулевых векторов: $\varphi_0(x) = x - \Phi_0(x)$, $\varphi_1(x) = x - \Phi_1(x)$. Говорят, что поля φ_0, φ_1 *гомотопны*, $\varphi_0 \simeq \varphi_1$, если существует вполне непрерывный оператор $\Phi: S_\infty \times I \to X$, $I = [0,1]$, удовлетворяющий условиям:

$$\varphi_t(x) = x - \Phi(x,t) \neq 0: \quad x \in S_\infty. \quad 0 \leqslant t \leqslant 1:$$

$$\Phi(x,0) = \Phi_0(x), \quad \Phi(x,1) = \Phi_1(x); \quad x \in S_\infty.$$

Если $\varphi_0 \simeq \varphi_1$, то $\chi(\varphi_0, S_\infty) = \chi(\varphi_1, S_\infty)$ [7].

В дальнейшем будем пользоваться следующим предложением:

Теорема Красносельского-Борсука *). *Если на единичной сфере банахова пространства* X *задано исчезающее вполне непрерывное векторное поле* $\varphi: S^0_\infty \to X$ *удовлетворяющее условию нечётности:* $\varphi(x) = -\varphi(-x)$ *для любого* $x \in S^0_\infty$, *то его характеристика* $\chi(\varphi, S^0_\infty)$ *нечетна.*

Основной результат

3. Лемма. *Пусть непрерывное отображение* $f: \overline{V}^0_\infty \to X$ *заданное на замыкании единичного шара* V^0_∞ *представлено в виде* (1). *Если* f *есть* ε*–отображение с постоянной* ε *равной единице, то степень* $d(f, V^0_\infty, f(0))$ *отображения* f *в точке* $f(0)$ *отлична от нуля.*

Доказательство. Из условия леммы вытекает, что $f(0) \in f(S^0_\infty)$ и степень $d(f, V^0_\infty, f(0))$ отображения f в точке $f(0)$ равна характеристике $\chi(\varphi, S^0_\infty)$

*) Эта теорема представляет собой обобщение на бесконечномерный случай известной теоремы К. Борсука об антиподах (см. [3] стр. 178). Эта же теорема и ее доказательство являются составной частью доказательства несколько иной теоремы в работе М. Красносельского [5].

векторного поля q определенного на сфере S^0_∞ по формуле:

$$(2)\qquad \varphi(x)=f(x)-f(0)=x-F(x)+F(0);\quad \varphi(x)\neq 0,\quad x\in S_\infty.$$

Для любой точки $x\in S^0_\infty$ положим $x_t^{(1)}=\frac{x}{1+t}$, $x_t^{(2)}=-\frac{tx}{1+t}$, где $0\leqslant t\leqslant 1$. При любом значении параметра $t(0\leqslant t\leqslant 1)$ точки $x_t^{(1)}$, $x_t^{(2)}$ принадлежат, очевидно, замкнутому шару $\overline{V^0_\infty}$, причем их расстояние $\|x_t^{(1)}-x_t^{(2)}\|=1$. Из условия леммы мы имеем поэтому $f(x_t^{(1)})\neq f(x_t^{(2)})$.

Определим теперь на сфере S^0_∞ семейство $\varphi_t(x)$ $x\in S^0_\infty$. $0\leqslant t\leqslant 1$ вполне непрерывных векторных полей по формуле:

$$(3)\qquad \varphi_t(x)=f(x_t^{(1)})-f(x_t^{(2)})=x-\left[F\left(\frac{x}{1+t}\right)-F\left(\frac{-tx}{1+t}\right)\right].$$

Семейство φ_t в силу определения (3) не имеет на S^0_∞ нулевых векторов, а вполне непрерывный при любом t оператор $\Phi(x,t)=\left[F\left(\frac{x}{1+t}\right)-F\left(\frac{-tx}{1+t}\right)\right]$, в силу равномерной непрерывности оператора F на $\overline{V}^0_\infty$, равномерно непрерывен по t при $x\in S^0_\infty$. Отсюда следует, что поля φ_0 и φ_1 гомотопны, а значит их характеристики одинаковы.

В силу (3) мы имеем:

$$\varphi_0(x)=x-F(x)+F(0)=\varphi(x),$$
$$\varphi_1(x)=x-\left[F\left(\frac{x}{2}\right)-F\left(-\frac{x}{2}\right)\right].$$

Так как поле φ_1 нечетно, то в силу приведенной теоремы Красносельского-Борсука заключаем, что характеристика $\chi(\varphi, S^0_\infty)$ поля $\varphi=\varphi_0$ отлична от нуля и лемма, таким образом, доказана.

4. Теорема 1. *Пусть O — открытое множество расположенное в банаховом пространстве X. Если, представимое в виде* (1), *отображение $f: O\to X$ обладает на O свойством (a^*), то образ $f(O)$ множества O при отображении f является открытым множеством в X.*

Доказательство. Пусть $y_0=f(x_0)$ произвольная точка из $f(O)$. Покажем, что некоторая окрестность точки y_0 полностью содержится в $f(O)$. Рассмотрим замкнутый шар $\overline{V^1_\infty}=\overline{V^1_\infty(x_0,\varepsilon_{x_0})}$ обозначив через S^1_∞ его границу. В силу условия теоремы $\overline{V^1_\infty}\subset O$ и f рассматриваемое на $\overline{V^1_\infty}$ есть ε_{x_0} — отображение. Множество $f(S^1_\infty)$ замкнуто и так как из условия теоремы следует, что $y_0\bar{\in} f(S^1_\infty)$, то число $\varrho(x_0)=\inf\|y_0-f(x)\|$, $x\in S^1_\infty$ положительно. Из доказанной леммы следует, что степень $d(f,V^1_\infty,f(x_0))$ отображения f в точке $y_0=f(x_0)$ отлична от нуля, а из определения числа $\varrho(x_0)$ следует, что степень

$d(f, V_\infty^1, y)$ отображения f для любой точки y удовлетворяющей неравенству $\|y - y_0\| < \varrho(x_0)$ отлична от нуля. Отсюда заключаем, что любая точка y принадлежащая $\varrho(x_0)$ — окрестности точки y_0 является образом некоторой точки $x \in \overline{V_\infty^1} \subset O$ при отображении f.

Итак, если некоторая точка $y_0 \in f(O)$, то достаточно малая окрестность точки y_0 (именно $\varrho(x_0)$ — окрестность) полностью содержится в $f(O)$, т. е. множество $f(O)$ открыто в пространстве X. Следовательно, теорема доказана.

Следствия

5. Из теоремы 1 непосредственно вытекает следующая:

Теорема 2. *Пусть непрерывное отображение f банахова пространства X в себя обладает следующим свойством: для любой точки $x \in X$ существует положительное число ε_x такое, что f рассматриваемое на замкнутом шаре $\overline{V_\infty(x, \varepsilon_x)}$ есть ε_x — отображение. Если f представимо в виде* (1), *то образ $f(X)$ пространства X при отображении f является областью в X.*

Из теоремы 2 вытекает доказанная М. Альтманом в работе [1]:

Теорема 2′. *Пусть непрерывное отображение $f: X \to X$ обладает следующим свойством: для любой точки $x \in X$ существует число $\varepsilon_x > 0$ такое, что f рассматриваемое на $\overline{V_\infty(x, \varepsilon_x)}$ есть ε_x — отображение в узком смысле. Если f представимо в виде* (1), *то $f(X)$ есть область в X.*

6. Из теоремы 2 вытекает, кроме того, следующая:

Теорема 3. *Пусть непрерывное отображение $f: X \to X$ банахова пространства X в себя есть ε — отображение. Если f представимо в виде* (1) *то множество $f(X)$ является областью в X.*

Из последней теоремы вытекает доказанная в работе [4]:

Теорема 3′. *Пусть непрерывное отображение $f: X \to X$ банахова пространства X в себя является ε — отображением в узком смысле. Если f представимо в виде* (1), *то пространство X переходит при отображении f само на себя, т. е. $f(X) = X$.*

Отметим, что теоремы 3 и 3′ являются обобщением на бесконечномерный случай теоремы К. Борсука [2], касающейся непрерывных отображений евклидовых пространств.

7. Из теоремы 1 вытекаем следующая:

Теорема 4. *Пусть O есть открытое множество в X, а непрерывное отображение $f: O \to X$ представлено в виде* (1). *Если f локально взаимнооднозначно, то множество $f(O)$ открыто в X.*

Частным случаем последней теоремы является теорема об инвариантности области в банаховом пространстве:

Теорема Шаудера-Брауэра. *Пусть O есть открытое множество расположенное в X, а непрерывное отображение $f: O \to X$ представимо в виде* (1). *Если f взаимно-однозначно, то множество $f(O)$ открыто в X.*

УНИВЕРСИТЕТ ИМ. НИКОЛАЯ КОПЕРНИКА, ТОРУНЬ

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

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[8] J. Schauder, *Invarianz des Gebiets in Funktionalräumen*, Studia Math., **1** (1929).

A. GRANAS, ON CONTINUOUS MAPPINGS OF OPEN SETS IN BANACH SPACE

Let X be a Banach space. We say that a continuous function $f: X_1 \to X$, $X_1 \subset X$ is an *ε-mapping* ($\varepsilon > 0$), if

$$(\alpha) \qquad [f(x_1) = f(x_2)] \to [\|x_1 - x_2\| < \varepsilon], \qquad x_1, x_2 \in X_1.$$

If, however, there exist two constants $\eta > 0$ and $\varepsilon > 0$ such that

$$(\beta) \qquad [\|f(x_1) - f(x_2)\| < \eta] \to [\|x_1 - x_2\| < \varepsilon], \qquad x_1, x_2 \in X_1,$$

then f is called an *ε-mapping in the narrow sense*. Let O be an open set contained in X. We put

$$V_\infty(x_0, \varrho) = \underset{x \in X}{E}[\|x_0 - x\| < \varrho], \quad S_\infty(x_0, \varrho) = \underset{x \in X}{E}[\|x - x_0\| = \varrho].$$

Theorem. *Let the mapping $f: O \to X$ satisfy the following condition:*

For each $x \in O$ there exists a number $\varepsilon_x > 0$, such that $V_\infty(x, \varepsilon_x) \subset O$, and f is an ε_x-mapping over the set $V_\infty(x, \varepsilon_x)$. If f is of the form $f(x) = x - F(x)$, where $F: O \to X$ is a completely continuous operator, then $f(O)$ is an open set in X.

From this theorem we obtain the Brouwer-Schauder Invariance of Domain Theorem [8] and the following

Theorem. *Let us assume that the mapping $f: X \to X$ of the Banach space into itself is of the form $f(x) = x - F(x)$, where $F: X \to X$ is a completely continuous operator. If f is an ε-mapping, then $f(X)$ is an open set contained in X; if, however, f is an ε-mapping in the narrow sense, then $f(X) = X$.*

In the finite dimensional case this theorem gives the well-known Borsuk theorem on continuous mappings of Euclidean space [2]. The second part of this theorem was proved in [4].

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MATHÉMATIQUE

Sur l'existence de solutions de certains problèmes aux limites relatifs à un système d'équations différentielles hyperboliques

par

Z. SZMYDT

Présenté par T. WAŻEWSKI le 27 Novembre 1957

Dans la note [3] nous avons introduit deux nouveaux problèmes aux limites relatifs au système de n équations différentielles

$$(1) \qquad u^{(i)}_{xy} = f^{(i)}(x, y, u^{(1)}, \ldots, u^{(n)}, u^{(1)}_x, \ldots, u^{(n)}_x, u^{(1)}_y, \ldots, u^{(n)}_y) \qquad (i = 1, \ldots, n)$$

et dans les articles [4] et [5] nous avons démontré quelques théorèmes concernant l'existence d'une solution de ces problèmes. Dans la présente note nous allons résoudre d'une manière uniforme les deux problèmes en question. Nous obtiendrons en même temps une généralisation de certains de nos résultats antérieurs.

Notations et définitions. Soient Γ et Λ deux courbes continues

$$y = \gamma(x) \quad \text{lorsque} \quad -\alpha \leqslant x \leqslant \alpha,$$
$$x = \lambda(y) \quad \text{lorsque} \quad -\beta \leqslant y \leqslant \beta$$

situées dans le rectangle D

$$-\alpha \leqslant x \leqslant \alpha \quad \text{et} \quad -\beta \leqslant y \leqslant \beta \quad \text{où} \quad 0 < \alpha < \infty \quad \text{et} \quad 0 < \beta < \infty$$

et soit (x_0, y_0) un point appartenant à D. Désignons par h le nombre

$$(2) \qquad h = 2\max(\alpha, \beta)$$

et par $U = (\mathring{u}^{(1)}, \ldots, \mathring{u}^{(n)})$ un vecteur constant.

Soit

$$U = (u^{(1)}, \ldots, u^{(n)}), \quad P = (p^{(1)}, \ldots, p^{(n)}), \quad Q = (q^{(1)}, \ldots, q^{(n)}), \quad F = (f^{(1)}, \ldots, f^{(n)}).$$

Considérons l'ensemble Π_1:

$$-\alpha \leqslant x \leqslant \alpha, \quad -\beta \leqslant y \leqslant \beta, \quad U, P \text{ et } Q \text{ arbitraires},$$

l'ensemble Σ_1:

$$-a \leqslant x \leqslant a, \quad U \text{ et } Q \text{ arbitraires},$$

et l'ensemble Σ_2:

$$-\beta \leqslant y \leqslant \beta, \quad U \text{ et } P \text{ arbitraires}.$$

Définition 1. Dans cette note la fonction $U(x, y)$ sera dite régulière dans le rectangle D, si elle est continue et admet des dérivées U_x, U_y, U_{xy} continues dans D.

Définition 2. Si $V = (v^{(1)}, \ldots, v^{(n)})$ est un vecteur à n dimensions, alors $|V| = \max\limits_{1 \leqslant i \leqslant n} |v^{(i)}|$.

Hypothèse B. 1° *Les fonctions* $F(x, y, U, P, Q) = (f^{(1)}(x, y, U, P, Q), \ldots, f^{(n)}(x, y, U, P, Q))$, $G(x, U, Q) = (g^{(1)}(x, U, Q), \ldots, g^{(n)}(x, U, Q))$ *et* $H(y, U, P) = (h^{(1)}(y, U, P), \ldots, h^{(n)}(y, U, P))$ *sont continues dans les ensembles* Π_1, Σ_1 *et* Σ_2 *respectivement.*

2° *Il existe des nombres finis* a, a_1, a_2, b, K_1, K_2 *et* d *tels que*

$$|F(x, y, U, P, Q)| \leqslant a|U| + b(|P| + |Q|) + d \quad \text{dans l'ensemble } \Pi_1, \tag{3}$$

$$|G(x, U, Q)| \leqslant a_1|U| + K_1|Q| + d \quad \text{dans l'ensemble } \Sigma_1, \tag{4}$$

$$|H(y, U, P)| \leqslant a_2|U| + K_2|P| + d \quad \text{dans l'ensemble } \Sigma_2. \tag{5}$$

Théorème 1. *Admettons l'hypothèse B. les fonctions F, G, H étant assujetties aux conditions de Lipschitz:*

$$|F(x, y, U, P, Q) - F(x, y, U, \widetilde{P}, \widetilde{Q})| \leqslant M(|P - \widetilde{P}| + |Q - \widetilde{Q}|) \quad \text{dans l'ensemble } \Pi_1, \tag{6}$$

$$|G(x, U, Q) - G(x, U, \widetilde{Q})| \leqslant K_1|Q - \widetilde{Q}| \quad \text{dans l'ensemble } \Sigma_1, \tag{7}$$

$$|H(y, U, P) - H(y, U, \widetilde{P})| \leqslant K_2|P - \widetilde{P}| \quad \text{dans l'ensemble } \Sigma_2$$

pour des constantes hM (cf. (2)) *et* K_1K_2 *satisfaisant aux inégalités:*

$$hM < 1 \quad \text{et} \quad K_1K_2 < (1 - hM)^2.$$

Alors, dans l'hypothèse que les constantes hb, ha, ha_1 et ha_2 soient suffisamment petites *) il existe une solution regulière $U(x, y)$ du système (1) dans le rectangle D satisfaisant aux conditions:

$$U(x_0, y_0) = \breve{U}, \tag{8}$$

$$U_x(x, y) = G[x, U(x, y), U_y(x, y)] \quad \text{lorsque } y = \gamma(x) \text{ et } -a \leqslant x \leqslant a, \tag{9}$$

$$U_y(x, y) = H[y, U(x, y), U_x(x, y)] \quad \text{lorsque } x = \lambda(y) \text{ et } -\beta \leqslant y \leqslant \beta. \tag{10}$$

*) Dans le cas $a_1 = a_2 = 0$, $a = b$, $h = 1$, il suffit d'admettre $b(3 + 2K_1 + 2K_2 + K_1K_2) < 1 - K_1K_2$.

Dans la démonstration du théorème 1, basée sur le théorème de J. Schauder sur le point invariant ([2], Satz 1, p. 173) nous nous appuierons sur le lemme 1 (démontré dans le § 2 de [5]) et sur le lemme 2, qui suivent.

LEMME 1. *Supposons que les fonctions F, G, H soient continues dans les ensembles Π_1, Σ_1 et Σ_2 respectivement.*

Alors l'existence d'une solution régulière $U(x, y)$ du système (1) *dans le rectangle D satisfaisant aux conditions* (8)-(10) *résulte de celle d'un point fixe de la transformation*

$$\begin{aligned}(11)\qquad \tilde{U}(x,y) = L[U] \equiv U &+ \int_{x_0}^{x} G[s, U(s, \gamma(s)), U_y(s, \gamma(s))]\,ds + \\ &+ \int_{y_0}^{y} H[t, U(\lambda(t), t), U_x(\lambda(t), t)]\,dt + \\ &+ \int_{x_0}^{x} \Big\{ \int_{\gamma(s)}^{y_0} F[s, t, U(s,t), U_x(s,t), U_y(s,t)]\,dt \Big\} ds + \\ &+ \int_{y_0}^{y} \Big\{ \int_{\lambda(t)}^{x} F[s, t, U(s,t), U_x(s,t), U_y(s,t)]\,ds \Big\} dt\,,\end{aligned}$$

définie dans un sous-ensemble de l'espace C^1 des fonctions $U(x, y)$ définies dans le rectangle D.

LEMME 2. *Admettons l'hypothèse B.*

Alors, dans l'hypothèse que $K_1 K_2 < 1$ et que les constantes hb, ha, ha_1 et ha_2 (cf. (2)) *soient suffisamment petites, il existent des nombres positifs r, r_1 et r_2 tels que la transformation* (11) *transforme l'ensemble des fonctions $U(x, y)$ de classe C^1 dans le rectangle D et y satisfaisant aux conditions:*

$$|U(x,y)| \leqslant r, \qquad |U_x(x,y)| \leqslant r_1, \qquad |U_y(x,y)| \leqslant r_2$$

en un sous-ensemble de celui-ci *).

Nous omettrons ici la démonstration du lemme 2, facile à reconstruire.

Choisissons arbitrairement les nombres r, r_1 et r_2 ayant les propriétés énumérées dans le lemme 2 et soit

$$N = ar + b(r_1 + r_2) + d\,.$$

Introduisons auxiliairement les ensembles suivants:

$$(12)\quad -a \leqslant x \leqslant a, \quad -\beta \leqslant y \leqslant \beta, \quad |U| \leqslant r, \quad |P| \leqslant r_1 \quad \text{et} \quad |Q| \leqslant r_2,$$

$$(13)\quad -a \leqslant x \leqslant a, \quad |U| \leqslant r \quad \text{et} \quad |P| \leqslant r_1,$$

$$(14)\quad -\beta \leqslant y \leqslant \beta, \quad |U| \leqslant r \quad \text{et} \quad |Q| \leqslant r_2.$$

*) Dans le cas où $a = b = a_1 = a_2 = 0$ on peut admettre $r_1 = d(1+K_1)(1+h)/(1-K_1K_2)$, $r_2 = d(1+K_2)(1+h)/(1-K_1K_2)$, $r = |\mathring{U}| + h(r_1 + r_2)$.

Soit $\delta(\varepsilon)$ le module de continuité ([3], p. 70, renvoi) commun à la fonction γ dans l'intervalle $a \leqslant x \leqslant a$, à la fonction λ dans l'intervalle $-\beta \leqslant y \leqslant \beta$ et aux fonctions F, G, H dans les ensembles (12), (13) et (14) respectivement.

Désignons par $\delta_1(\varepsilon)$ et $\delta_2(\varepsilon)$ les fonctions non décroissantes tendant vers zéro avec ε:

$$(15)\qquad \begin{aligned} \delta_1(\varepsilon) &= (1+h+N)\delta(\varepsilon)+\delta(r_1\varepsilon+r_2\delta(\varepsilon))+(K_1+Mh)N\varepsilon+h\delta(r_1\varepsilon), \\ \delta_2(\varepsilon) &= (1+h+N)\delta(\varepsilon)+\delta(r_2\varepsilon+r_1\delta(\varepsilon))+(K_2+Mh)N\varepsilon+h\delta(r_2\varepsilon). \end{aligned}$$

On vérifie facilement que le système d'équations

$$(16)\qquad \begin{aligned} \delta_1(\varepsilon)+K_1\mu(\delta(\varepsilon))+Mh\varkappa(\varepsilon) &= \varkappa(\varepsilon), \\ \delta_2(\varepsilon)+K_2\varkappa(\delta(\varepsilon))+Mh\mu(\varepsilon) &= \mu(\varepsilon) \end{aligned}$$

possède en vertu de nos hypothèses une seule solution $\varkappa(\varepsilon)$, $\mu(\varepsilon)$ définie pour $\varepsilon \geqslant 0$, non-décroissante et tendant vers zéro avec ε.

Considérons dans l'espace C^1 l'ensemble Z des fonctions $U(x, y)$ définies dans le rectangle D et y satisfaisant aux inégalités

$$|U(x,y)| \leqslant r, \quad |U_x(x,y)| \leqslant r_1, \quad |U_y(x,y)| \leqslant r_2,$$

$$|U_x(x,\bar{y})-U_x(x,y)| \leqslant N|\bar{y}-y|, \quad |U_y(\bar{x},y)-U_y(x,y)| \leqslant N|\bar{x}-x|,$$

$$|U_x(\bar{x},y)-U_x(\bar{\bar{x}},y)| \leqslant \varkappa(|\bar{x}-\bar{\bar{x}}|), \quad |U_y(x,\bar{y})-U_y(x,\bar{\bar{y}})| \leqslant \mu(|\bar{y}-\bar{\bar{y}}|).$$

On vérifie facilement que l'ensemble Z est convexe et compact dans l'espace C^1 et que la transformation (11) y est continue. Afin d'établir notre théorème en s'appuyant sur le théorème de Schauder ([2], Satz 1, p. 173) et en tenant compte des lemmes 1 et 2, il ne reste donc qu'à démontrer que les inégalités

$$(17)\qquad |\tilde{U}_x(x,\bar{y})-\tilde{U}_x(x,y)| \leqslant N|\bar{y}-y|, \quad |\tilde{U}_y(\bar{x},y)-\tilde{U}_y(\bar{\bar{x}},y)| \leqslant N|\bar{x}-\bar{\bar{x}}|,$$

$$(18)\qquad |\tilde{U}_x(\bar{x},y)-\tilde{U}_x(\bar{\bar{x}},y)| \leqslant \varkappa(|\bar{x}-\bar{\bar{x}}|), \quad |\tilde{U}_y(x,\bar{y})-\tilde{U}_y(x,\bar{\bar{y}})| \leqslant \mu(|\bar{y}-\bar{\bar{y}}|)$$

sont satisfaites dans le rectangle D. Or les inégalités (17) résultent de (11) car $|F(x,y,U,P,Q)| \leqslant ar+b(r_1+r_2)+d = N$ dans l'ensemble (12). En profitant en plus des relations (15) et (16) on établit les inégalités (18). Le théorème 1 se trouve ainsi démontré.

Le théorème 2 que voici ([5], remarque 1)*) résulte immédiatement du théorème 1.

*) Le théorème 1 (théorème 2) assure l'existence d'une solution du problème I (problème II) considéré dans l'article [5].

THÉORÈME 2. *Soient* $F(x, y, U, P, Q)$ *et* $G(x, U, Q)$ *des fonctions continues dans les ensembles* Π_1 *et* Σ_1 *respectivement, qui y satisfont aux conditions* (3), (6) *et* (4), (7).

Soit $B(y, U, P)$ *une fonction continue dans l'ensemble* Σ_2 *qui y satisfait aux conditions*

$$|B(y, U, P) - B(y, U, \widetilde{P})| \leqslant K^*|P - \widetilde{P}|,$$
$$|B(y, U, P)| \leqslant K^*|P| + a_2|U| + d.$$

Supposons que $\lambda(y)$ *soit une fonction de classe* C^1 *dans l'intervalle* $[-\beta, \beta]$ *et que* (cf. (2), (6), (7))

$$hM < 1, \qquad K_1(K^* + w) < (1 - hM)^2 \qquad où \qquad w = \max_{-\beta \leqslant y \leqslant \beta} |\lambda'(y)|.$$

Alors, dans l'hypothèse que les constantes hb, ha, ha_1, ha_2 *soient suffisamment petites, le système* (1) *admet une solution régulière* $U(x, y)$ *dans le rectangle* D *satisfaisant aux conditions*

$$U_x(x, y) = G[x, U(x, y), U_y(x, y)] \quad \text{lorsque} \quad y = \gamma(x) \quad \text{et} \quad -\alpha \leqslant x \leqslant \alpha,$$

$$U(\lambda(y), y) = U - \int_{y_0}^{y} B[t, U(\lambda(t), t), U_x(\lambda(t), t)]\, dt \quad \text{lorsque} \quad -\beta \leqslant y \leqslant \beta.$$

Remarque 1. Il est évident que les théorèmes 1 et 2 dans lesquels $a = a_1 - a_2 = b = 0$ généralisent cette partie de la remarque 5 de l'article [5] qui se rapporte aux théorèmes 2 et 4 de [5]. En utilisant le lemme 2 on peut établir aussi le théorème généralisant cette partie de la remarque en question qui se rapporte au théorème 1 de [5]. Il en suit que dans le résultat de l'article [5] mentionné ci-dessus, l'hypothèse que les fonctions G et H soient bornées dans les ensembles Σ_1 et Σ_2 respectivement ([5], § 7, hypothèse R_I) peut être remplacée par une hypothèse moins restrictive, à savoir par celle que les inégalités (4) et (5) avec $a_1 = a_2 = 0$, $K_1 K_2 < 1$ soient satisfaites.

Remarque 2. Dans le cas du problème de Darboux, auquel correspond dans nos notations celui où $\gamma(x) \equiv y_0$, $\lambda(y) \equiv x_0$, G ne dépend que de x, H ne dépend que de y, l'hypothèse (3) a été considéré par C. Ciliberto [1].

Remarque 3. En appliquant un procédé analogue à celui qui nous a permis de démontrer le théorème 1, on arrive au théorème, qui généralise cette partie de la remarque 1 de la note [4] qui se rapporte au théorème 2 de [4]. On constate en particulier que l'hypothèse A^{bis} 2° admise dans [4] peut être remplacée par une hypothèse moins restrictive, à savoir par celle que les inégalités (4) et (5) avec $a_1 = a_2 = 0$ soient satisfaites.

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Deuteron Stripping on Spheroidal Nuclei. I

by

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1. Introduction

The deuteron stripping effect is well known as one of the best tools of the modern nuclear spectroscopy. Up-to-date analyses of this effect were confined to the application of the nuclear shell model, pure or taking into account the configuration mixing of shells. In the latter case only perturbation methods have been used. The collective aspects were taken into account only by Satchler [1], who discussed reduced widths for capture to vibrational and rotational states using also the theory of perturbations. The configuration mixing effect (two or more l_n-values effective in stripping angular distributions) has been explained for medium heavy nuclei in terms of perturbation methods using two-body L-S or intermediate couplings (see, e. g. Bethe and Butler [2], Okai and Sano [3] — for relative reduced widths e. g. Auerbach and French [4] for angular correlations with γ-s following stripping or pick-up reactions e. g. Satchler [5], Hittmair [6]). The perturbation theory of the configuration mixing coefficients seems neither consistent with the basic (shell) model assumed nor does its validity seem obvious.

The best known target nuclei in question are rather strongly deformed as lying in the 8-20 region. It seems, therefore, that the spheroidal nuclear model able to represent the shell configuration mixing consistently may well serve as an explanation of the effects in question. The directly calculated mixing coefficients are perfectly consistent with the nuclear model, since they automatically follow from this model. S. G. Nilsson [7] presented energy spectra and one particle wave functions for a simple spheroidal harmonic oscillator well. In the present paper S. G. Nilsson's wave functions are employed, and the modification for the, perhaps more realistic but more complicated, case of Gottfried's [8] elipsoidal model is straightforward.

In Section 2A general expressions for the angular distributions and the relative differential cross-sections for transitions to different final states are given. Numerical examples are discussed in Section 2B.

2. Angular distributions

A. General Theory. Let the symmetrized wave function of the ground state of the target nucleus of spin I_i projection on the symmetry (intrinsic nuclear) axis $z' = K_i$ on the z-axis $- M_i$ be in the usual notation [7]:

$$(1)\quad \Psi_i = \sqrt{\frac{2I_i+1}{16\pi^2}}\,\Phi_0(\xi)[\chi_{\Omega_i}(i')D^{I_i}_{M_iK_i}(\theta_i) + (\quad)^{I_i-\Sigma j_i}\chi_{-\Omega_i}(i')D^{I_i}_{M_i-K_i}(\theta_i)]\,.$$

where $\Phi_0(\xi)$ is the spinless core wave function of the vibrational ground state, $\chi_{\Omega_i}(i')$ – the wave function of the particles outside the core, being, in general, a product of S. G. Nilsson's one particle solutions; "i'" denotes the co-ordinates of all the particles in the intrinsic system; $D^I_{MK}(\theta_i)$ are the usual rotational matrices depending on the Eulerian angles θ_i. Let us consider a (d, p) reaction for definiteness. Let the function of the final nucleus of spin I_f and projection K_f and M_f be:

$$(2)\quad \Psi_f = \sqrt{\frac{2I_f+1}{16\pi^2}}\,\Phi_0(\xi)\times$$
$$\times[\chi_{\Omega_f}(i', n'_f)D^{I_f}_{M_fK_f}(\theta_i) + (-)^{I_f-\Sigma(j_t+j_n)}\chi_{-\Omega_f}(i', n'_f)D^{I_f}_{M_f-K_f}(\theta_i)]\,.$$

The parity and $j_z = l_z + s_z$ are the operators commuting with the total Hamiltonian. The j_z quantum number is, as usually, Ω. No vibrational excited states are considered, $n'_f s$-are all co-ordinates of the captured neutron. Let us now make the simplifying assumption characteristic for usual stripping calculations, i. e. neglect antisymmetrization in protons *). The Ω-value of the captured neutron, $\Omega_n = \Omega_f - \Omega_i$. Using the orthonormality of Φ_0 and χ_Ω's, one can obtain for the overlap integral:

$$(3)\quad \int d\xi di' \Psi^*_f \Psi_i = \sqrt{\frac{(2I_i+1)(2I_f+1)}{(16\pi^2)^2}}\{\chi^*_{\Omega_n}(n'_f)D^{I_f^*}_{M_iK_i}(\theta_i)D^{I_i}_{M_iK_i}(\theta_i) +$$
$$+(-)^{I_f-I_i-\Sigma j_n}\chi^*_{-\Omega_n}(n'_f)D^{I_f^*}_{M_f-K_f}(\theta_i)D^{I_i}_{M_i-K_i}(\theta_i)\}\,.$$

For S. G. Nilsson's (l, Λ)-representation the symmetrization operator $(-)^{I_f-I_i-\Sigma j_n}$ can be symbolically rewritten as $(-)^{I_f-I_i-1/2+l_n}$, l_n being the neutron orbital angular momentum.

*) For the sake of calculation of a (d, p) matrix element we may put $\chi_{\Omega_f}(i'n'_f) = \chi_{\Omega_i}(i')\chi_{\Omega_n}(n'_f)$.

For the sake of simplicity we shall first calculate the Butler type (d, p) matrix element, i. e. we shall replace the final state proton wave by $e^{i\vec{k}_p\vec{r}_p}\chi_{\mu_p}(\sigma_p)$ and the incident deuteron wave by $e^{i\vec{k}_d\vec{r}_d}S_{\mu_d}(\sigma_n, \sigma_p)$ $\varphi(\vec{r}_n - \vec{r}_p)$, χ and S being the usual spin functions; $\varphi(\varrho)$ is the deuteron internal function. The "distorted wave" generalization is straightforward in the zero range approximation. Then the matrix element is expressed by the usual notation:

$$(4)\quad I^{K_i}_{\Omega_n}(M_i\mu_a;\ M_f\mu_p) \int d\xi d\vec{i}' \int d\theta_i \int\limits_{r_n\geq r_0} d\vec{r}_n d\vec{r}_p d\sigma_n d\sigma_p \Psi^*_f \times$$

$$\times e^{-i\vec{k}_p\vec{r}'_p}\chi^*_{\mu_p}(\sigma_p) V_{np}(\vec{r}_n - \vec{r}_p)\,\Psi_i e^{i\vec{k}_d\vec{r}_d} S_{\mu_d}(\sigma_n, \sigma_p)\varphi(|\vec{r}_n - \vec{r}_p|);$$

V_{np} being the usual transition operator and r_0 – the (Butler) nuclear radius. We can now transform the captured neutron function from the intrinsic nuclear system to the fixed system, viz:

$$\chi_{\Omega_n}(n'_f) \to \chi_{\Omega_n}(n_f) = \sum_{l_n\Sigma} A_{l_n\Omega_n-\Sigma} R_{Nl_n}(r_n)\times$$

$$(5)\qquad \times \sum_{j_n\mu} (l_n\tfrac{1}{2};\ \Omega_n - \Sigma\Sigma\,|\,j_n\Omega_n) D^{j^*_n}_{\mu\Omega_n}(\theta_i)\mathcal{Y}^{\mu}_{j_n l_n \frac{1}{2}}(\omega_n, \sigma_n)\,.$$

Here

$$R_{Nl_n}(r_n) = N_{Nl_n} r^{l_n} e^{-\frac{1}{2}\Gamma r_n^2} L_n^{l_n+\frac{1}{2}}(\Gamma r_n^2)$$

and $\Gamma_n = \dfrac{M\omega_0(\delta)}{\hbar}$ [7]; $(j_1 j_2 m_1 m_2|jm)$ is a vector addition coefficient and $\mathcal{Y}^m_{jls}$ – a usual j-eigenfunction *) (compare [15]).

The final result becomes:

$$I^{K_i}_{\Omega_n}(M_i\mu_a;\ M_f\mu_p) \sim (\tfrac{1}{2}\tfrac{1}{2};\ \mu_p\mu_d - \mu_p|1\mu_d) \sum \{l_n j_n \Sigma_\mu\}$$

$$(6)\qquad (l_n\tfrac{1}{2};\ \mu - \mu_d - \mu_p\mu_d \quad \mu_p\ j_n\mu)(I_i j_n;\ M_i\mu_i\, I_f M_f)\, B_{I_f}(l_n j_n \Sigma|\Omega_n K_i)$$

$$S(l\mu - \mu_d + \mu_p)\int d\vec{\varrho}\, e^{i\vec{K}\vec{\varrho}} V_{np}(\varrho)\varphi(\varrho)\,,$$

where

$$\vec{K} = \frac{\vec{k}_d}{2} - \vec{k}_p,\ S(l_n, m) = \int\limits_{r_n\geq r_0} d\vec{r}_n e^{i\vec{\varkappa}\vec{r}_n} R_{Nl_n}(r_n)\, Y^*_{l_n m}(\omega_n),\ \vec{\varkappa} = \vec{k}_d - \frac{M_i}{M_p}\vec{k}_p\,,$$

*) The principal quantum number N is, as in [7], practically a constant of the motion.

M_i and M_f being the masses of the (i) and (f) nuclei:

$$(7)\quad B_{I_f}(l_n j_n \Sigma|\Omega_n K_i) = \{(l_n \tfrac{1}{2}; \Omega_n - \Sigma\Sigma|j_n\Omega_n)(I_i j_n;\ K_i\Omega_n|I_f K_f) A_{l_n\Omega_n-\Sigma} +$$
$$+ (-)^{I_f-I_i-\frac{1}{2}+l_n}(l_n\tfrac{1}{2};\ -\Omega_n+\Sigma-\Sigma|j_n-\Omega_n)(I_i j_n;\ -K_i-\Omega_n|I_f-K_f)\times$$
$$\times A_{l_n-\Omega_n+\Sigma}\} = 2A_{l_n\Omega_n-\Sigma}(l_n\tfrac{1}{2};\ \Omega_n-\Sigma\Sigma|j_n\Omega_n)(I_i j_n;\ K_i\Omega_n|I_f K_f)\,.$$

Then the differential cross-section can be shown, in general to be *)

$$(8)\quad \sigma(\theta)\sim \sum_{l_n j_n}(2l_n+1)^{-1}P_{I_f}(l_n j_n|\Omega_n K_i)\sum_m |S(l_n m)|^2 G(K)$$
$$P_{I_f}(l_n j_n|\Omega_n K_i) = \sum_{\Sigma\Sigma'} B_{I_f}(l_n j_n\,\Sigma|\Omega_n K_i)\,B_{I_f}(l_n j_n\,\Sigma'|\Omega_n K_i)\,.$$

If the quantization axis is chosen along the vector $\vec{\varkappa}$, we get the selection rule: $\mu = \mu_d - \mu_p$. Now $S(l_n m) = \delta_{0m}(2l_n+1)^{1/2}\varrho^{l_n}\mathcal{R}_{Nl_n}(\varkappa)$, where

$$\mathcal{R}_{Nl_n}(\varkappa) = \int_0^\infty j_{l_n}(|\vec{\varkappa}|r)R_{Nl_n}(r)r^2dr\,;$$

$j_k(x)$ being a spherical Bessel function, and:

$$(8a)\quad \sigma(\theta)\sim \sum_{l_n j_n} P_{I_f}(l_n j_n|\Omega_n K_i)\,|\mathcal{R}_{Nl_n}(\varkappa)|^2 G(K)\,;$$

if $\varphi(\varrho)$ be a Hulthén function of the parameters α_d and β_d, we have $G(K) = \left(1-\frac{K^2+\alpha_d^2}{K^2+\beta_d^2}\right)^2$ (see e. g. Toboeman [9]). The factor $G(K)$ does not appear in the zero-range approximation **). The functions $\mathcal{R}_{Nl_n}(\varkappa)^2$ give the characteristic stripping peaks of the angular distributions for various l_n as in the usual theory. Different l_n contribute incoherently to $\sigma(\theta)$ as in the usual theory.

An alternative derivation of Eqs. (6)-(8) can be easily performed by using the intrinsic nuclear system rather than the fixed one (i. e. by transforming all the functions, except Ψ_i and Ψ_f, to the intrinsic system).

*) In the j-representation one should have:

$$P_{I_f}(l_0 j_n|\Omega_n K_i) = 4(I_i j_n;\ K_i\Omega_n\ I_f K_f)^2\, C_{j_n l_n}(\Omega_n)^2 = 4\left(\frac{2I_f+1}{2I_i+1}\right)\beta_{j_n l_n}(K_i\Omega_n)^2,$$

where

$$C_{j_n l_n}(\Omega_n) = \sum_\Sigma (l_n\tfrac{1}{2};\ \Sigma\Sigma|\,j_n\Omega_n)\,A_{l_n\Omega_n-\Sigma}\,.$$

$\beta_{j_n l_n}(K_i\Omega_n)$ is a respectively reduced width provided $\sum_{l_n 1} A^2_{l_n 1} = 1$.

**) i. e. for βd . . .

We have the selection rules: $K_f = K_i + \Omega_n$ and $j_n \geq \Omega_n$. The allowed values of j_n are: $I_f - I_i \leq j_n \leq I_f + I_i$. If one I_i or I_f is zero, then only one-one value of l_n contributes, i. e. we have no "mixing effect". This is the case for even-even target nuclei with $I_i = K_i = \Omega_i = 0$ and for even-even final nuclei in their ground states with $I_f = K_f = \Omega_f = 0$ (i. e. $\Omega_n = -\Omega_i$). It should be noted here that the known (d, p) reactions with two or more $l_n's$ effectively involve odd A target nuclei and mostly odd-odd final nuclei (or even-even in excited states) in agreement with the selection rule stated above. These are: $_9F^{19}$ (d, p) $_9F^{20}$, $_{12}Mg^{25}$ (d, p) $_{12}Mg^{26}$ (first excited state), $_{13}Al^{27}$ (d, p) $_{13}Al^{28}$ (third excited state), $_{15}P^{31}$ (d, p) P^{32}, $_{17}Cl^{35}$ (d, p) $_{17}Cl^{36}$, $_{19}K^{41}$ (d, p) $_{19}K^{42}$, $_{21}Sc^{45}$ (d, p) $_{21}Sc^{46}$, $_{23}V^{51}$ (d, p) $_{23}V^{52}$.

The mixing coefficients $P_{I_f}(l_n j_n \Omega_n K_i)$ depend on the particular spheroidal model assumed *via* the distortion coefficients $A_{l_n \Lambda}$, being rather sensitive functions of the deformation parameter δ for $A \approx 25$. These coefficients and the geometrical factors appearing in P can, in general, explain, why only one l_n contributes effectively in one case and more $l_n's$ in another case. In particular, they can predict why a transition takes place to one final state with purely one l_n (or "almost purely" one l_n) while another final state of the same nucleus, for the same reaction, involves two $l_n's$ or more. Because of the sensitiveness of $A's$ to δ, the P-coefficients could provide a useful tool for investigating the spheroidal model parameters.

The relative values of $\sigma(\theta)'s$ corresponding to different final states "1" and "2", their targets being common at a certain angle θ_0, can be written as:

$$\frac{d\sigma_1}{d\sigma_2}\Big|_{\theta=\theta_0} = \frac{(k_p)_1}{(k_p)_2} \frac{\sum_{l_n j_n} P_{I_{f_1}}(l_n j_n \Omega_1 K_i) |\mathcal{R}_{Nl_n}(\varkappa_1)|^2 G(K_1)}{\sum_{l_n j_n} P_{I_{f_2}}(l_n j_n \Omega_2 K_i) |\mathcal{R}_{Nl_n}(\varkappa_2)|^2 G(K_2)}\Big|_{\theta=0} . \tag{9}$$

These relative cross-sections can be useful in the analysis of the model parameters, similarly to analogous expressions in the theory of Auerbach and French [4].

B. Examples. Let us consider several examples of (d, p) reactions on nuclei of $A \approx 25$-28. It is well known that, in this A-region, and given the incident deuteron energy considered below ≈ 8 MeV, the Coulomb effects do not affect much the angular distributions and may be neglected. On the other hand, however, S. G. Nilsson's spectra and wave functions require modification in this A-region and the nuclear deformations are not well known. In particular, the $s_{1/2}$ orbital, most important in this region, should be raised relatively to its neighbouring orbitals [10]. Furthermore, the $A_{l_n \Lambda}$-coefficients are rather sensi-

tive *) to the deformation parameter δ. These facts and the neglect of the nuclear distortion effects of the deuteron and proton waves make it unlikely that good quantitative fits to the data can be obtained. One can, however, obtain qualitative agreement with the data and derive conclusions not only as to the assignments of spins and Ω_n-values of the final nuclear states, but also as to the possible modifications of the model wave functions (i. e. the $A_{l_n\Lambda}$ — coefficients).

Let us first consider (d, p) reactions on the ^{27}Al target. The two most important proton groups correspond to the ground state ($Q = -5.494$ MeV) and to the 1.015 MeV level ($Q = 4.479$ MeV) of ^{28}Al. According to Sheline [11] the spin assignment should be: $I_f = 3+K = 3$ and $I_f = 3+$, $K_f = 2$, respectively. Since $I_i = K_i = 5/2$, we have $\Omega_n = -1/2$ and $\Omega_n = -1/2$ in these two cases, respectively. This is consistent with the neutron configuration

$$\nu[{}_8O^{16},\ d_{5/2}(1/2^+{}^2,\ 3/2^+{}^2,\ 5/2^+{}^2),\ S_{1/2}(1/2^+{}^1)]$$

in both cases in question. Using Eq. (8) we find for the ground state:

$$\text{(10a)}\qquad P_3(0\tfrac{1}{2}|\tfrac{1}{2}\tfrac{5}{2}) = 4A_{00}^2,\ \sum_j P_3(2j|\tfrac{1}{2}\tfrac{5}{2}) = \frac{5}{3}A_{20}^2+\frac{35}{18}A_{21}^2-\frac{5\sqrt{6}}{9}A_{20}A_{21}$$

and for the excited state:

$$\text{(10b)}\qquad P_3(0\tfrac{1}{2}|-\tfrac{1}{2}\tfrac{5}{2}) = \tfrac{2}{3}A_{00}^2,\ \sum_j P_3(2j|-\tfrac{1}{2}\tfrac{5}{2}) = \tfrac{5}{3}(A_{20}^2+A_{2-1}^2)\,.$$

The most realistic deformation value of ^{28}Al seems to be $\beta \approx 0.20$ [12]. If S. G. Nilsson's functions are used for $\eta = 4$, we have:

$$a_g = \frac{P_3(0\tfrac{1}{2}|\tfrac{1}{2}\tfrac{5}{2})}{\sum_j P_3(2j|\tfrac{1}{2}\tfrac{5}{2})} \approx 0.78$$

and

$$a_e = \frac{P_3(0\tfrac{1}{2}|-\tfrac{1}{2}\tfrac{5}{2})}{\sum_j P_3(2j|-\tfrac{1}{2}\tfrac{5}{2})} \approx 0.155\,.$$

In the first (ground state) case an excellent fit to the data of Holt and Marsham [13] can be obtained by using $r_0 = 6.15\times 10^{-13}$ cm. The resulting angular distribution is very similar to theirs in Fig. 7. With $a_g \approx 0.78$ the $l_n = 2$ contribution is almost completely masked and the case is almost pure $l_n = 0$. As to the excited state, the $l_n = 2$ peak is very distinct. However, the excellent fit to the data (a curve closely similar to that of Holt and Marsham's of Fig. 8) would require $a_e \approx 0.09$

*) For $A \approx 25$-28.

rather than 0.155. A modification of $A's$ for the $s_{1/2}$ shell would be needed, because it is little probable that a "flattening" of the curves due to Coulomb effects would be sufficient to account for the discrepancy.

In cases of the 32 keV-level of ^{27}Al we should have an admixture of $l_n = 2$ to $l_n = 0$ according to our model and Sheline's spin assignment. The 2.198 MeV-level is a pure $l_n = 2$ case in agreement with the assumption $I_f = 2-$, $K_f = 0$. However, the 2.137 MeV-level with its $l_n = 0$ cannot be $I_f = 2+$, $K_f = 1$, as proposed by Sheline. Furthermore, if the 2.268 MeV-level is $I_f = 4-$, $K_f = 2$, it should be pure $l_n = 2$. Further, more accurate experiments seem to be indicated.

Consider now the ^{25}Mg target with $I_i = K_i = 5/2$. Consider two proton groups corresponding to the ground and the 1.825 MeV first excited levels of ^{26}Mg. The ground state is $I_f = 0+$, $K_f = 0$ (even-even nucleus) and the 1.825 MeV-state should be $I_f = 2$ (or 3)+, $K_f = 2$ $(l_n = 0) + (l_n = 2)$. The neutron configurations are supposed to be:

$$p[{}_8\mathrm{O}^{16}, d_{5/2}(1/2+^2, 3/2+^2, 5/2+^2)]$$

and

$$p[{}_8\mathrm{O}^{16}, d_{5/2}(1/2+^2, 3/2+^2, 5/2+^1)S_{1/2}(1/2+^1)],$$

respectively *). The ground state is pure $l_n = 2$ in agreement with the data and consistently with $I_f = K_f = 0$ and with the $\Omega_n^{\pi} = 5/2$ S. G. Nilsson's function. If it is now assumed that $I_f = 2$ then:

$$P_2(0\tfrac{1}{2}|-\tfrac{1}{2}\tfrac{5}{2}) = {}^{1,0}_{3}A^{\ 2}_{00}, \qquad \sum_j P_2(2j|-\tfrac{1}{2}\tfrac{5}{2}) = {}^{2\ 5}_{2\ 2}A^2_{20} + {}^{1,0}_{\ 1}A^2_{2-1} - \frac{10\sqrt{6}}{21}A_{20}A_{2-1}.$$

The deformation value comes from Sheline's [11] paper: $\delta \approx 0.363$. Unfortunately S. G. Nilsson's parameters are not known in this case. Furthermore, S. G. Nilsson's spectrum (Fig. 5 of [7]) for ^{26}Mg does not agree with the experimental evidence. If one takes $\eta_f = 6$ arbitrarily, one finds $a \approx 0.35$. On taking $r_0 = 5.27 \times 10^{-13}$ cm. after Okai and Sano [3] a good fit to Holt and Marsham's [14] data for $a \approx 0.082$ ($a \approx 0.092$ for $r_0 = 6.15 \times 10^{-13}$ cm.) can be obtained. If I_f were 3 rather than 2, we should find (as for ^{28}Al*) $a \approx 0.064$ for $\eta_f = 6$. To decide between $I_f = 2$ and $I_f = 3$ the $A's$ for the actual deformation $\eta_f \approx 7.3$. have to be known first. Okai and Sano [3] favour $I_f = 2$ rather than 3.

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*) This level may, however, be regarded as a 2+ member of the rotational band based on $K = 0$. This would require a different explanation of the experimental angular distribution involving a configuration mixing both in the target and in the final nucleus.

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Deuteron Stripping on Spheroidal Nuclei. II

by
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Presented by L. INFELD on October 25, 1957

1. Angular correlations with g-rays

It is well known that angular correlations with γ-rays, following a deuteron stripping reaction with the particle capture to an excited state of the final nucleus, provide useful tools in nuclear spectroscopy (they are sensitive to various coupling schemes – see, e. g., [2], [3] and also [4], [5]). In particular, it is well known that such correlations are especially useful in determining nuclear spins which, in general, can hardly be determined by angular distributions only.

It is well known that, if the neutron in a (d, p) reaction is captured with a definite orbital angular momentum l_n and a definite total angular momentum j, then the angular correlation function is just the product of the stripping cross-section with a factor depending on the angle of emission of the γ-ray, provided there is no distortion of the deuteron and/or proton wave. On the other hand, if two or more l_n's and j_n's are involved, there are, in general, not only mixtures of products of different factors, but also interference terms between the contributions of different l_n's and j_n's in the correlation function. This occurs even if no distortion of the deuteron and proton waves is assumed. Since, in our theory [1], mixtures of l_n's are of greatest interest, the angular correlations with γ's are the more interesting.

The statistical matrix for the emission of 2^L-pole radiation in the direction of propagation $\vec{\gamma}$ from a nucleus making a transition from a state of spin I_f to one of spin I_f' is proportional to (see [6]):

$$(1) \qquad W_{M_f M_f'} = \sum_{\lambda\nu} (2\lambda I_f; \nu M_f | I_f M_f')(LL; 1{-}1 | 2\lambda 0) W(I_f I_f LL; 2\lambda I_f') Y_{2\lambda\nu}(\vec{\gamma})^*,$$

where $W(j_1 j_2 j_3 j_4; I_1 I_2)$ is a usual Racah coefficient. If $U_{M_f M_f'}$ is the statistical matrix for the orientation of the residual nucleus in a stripping reaction, the correlation function is given as usually by:

$$(2) \qquad \Phi(\vec{k}_d, \vec{k}_p, \vec{\gamma}) = \sum_{M_f M_f'} U_{M_f M_f'}(\vec{k}_d, \vec{k}_p) W_{M_f M_f'}(\vec{\gamma}) .$$

Using the usual Racah sum rule technique, one can arrive at the following final result:

$$(3)\qquad \Phi \sim \sum_{\lambda\nu} Y_{2\lambda\nu}(\vec{\gamma}) \sum \{ll'jj'mm'\}\, \xi_{jj'}^{lml'm'}(\Omega_n K_i I_f)(-)^l \times$$
$$\times \sqrt{\frac{(2j+1)(2j'+1)}{2l+1}}\,(2\lambda l';\ \nu m'|lm) F_{2\lambda}(LI_f' I_f) \times$$
$$\times W(I_f I_f jj';\ 2\lambda I_i) W(ll'jj';\ 2\lambda \tfrac{1}{2}),$$

where

$$F_{2\lambda}(LI_f' I_f) = (-)^{I_f - I_f' - 1}(2I_f+1)^{1/2}(2L+1)(LL;\ 1\ -1|2\lambda 0) W(I_f I_f LL;\ 2\lambda I_f').$$

(tabulated in [6]), and

$$(4)\qquad \xi_{jj'}^{lml'm'}(\Omega_n k_i I_f) = P_{I_f}(ll'jj'|\Omega_n K_i)\, S(lm)\, S(l'm')^*,$$
$$P_{I_f}(\ldots) = 4 \sum_{\Sigma\Sigma'} (l\tfrac{1}{2};\ \Omega_n - \Sigma\Sigma|j\Omega_n)(l'\tfrac{1}{2};\ \Omega_n - \Sigma'\Sigma'|j'\Omega_n) \times$$
$$\times (I_i j;\ K_i \Omega_n | I_f K_f)(I_i j';\ k_i \Omega_n | I_f k_f) A_{l\Omega_n - \Sigma} A_{l'\Omega_n - \Sigma'}\ ^*),$$

and the factor $(-)^l$ can be omitted in (3) due to the parity conservation. The integrals $S(l,\ m)$ can be written, in general, in the zero range approximation as:

$$(5)\qquad S(lm) = \int\limits_{r \geqslant r_0} d\vec{r}\, F_{\vec{k}_p}^{*}(\vec{r}) F_{\vec{k}_d}^{*}(\vec{r})\, R_{Nl}(r)\, Y_{lm}^{*}(\Omega'),$$

$F_{\vec{k}_d}$ and $F_{\vec{k}_r}$ being the deuteron and proton waves distorted by a nonspin dependent potential (the most realistic distortion potential for the proton would probably be a spheroidal optical model well, as given in [7], [8]). If the quantization axis is chosen along the vector k and the x-axis lies in the reaction plane, we can write:

$$(6)\qquad S(lm) \sim \sum_{L_p L_d} P_{L_p}^{m}(\cos\theta_{\vec{k}_p})\, i^{L_d - L_p} \sqrt{\frac{(2L_p+1)(2L_d+1)}{2l+1}}\, \sqrt{\frac{(L_p-|m|)!}{(L_p+|m|)!}} \times$$
$$\times (L_p L_d;\ 00|l0)(L_p L_d;\ m0|lm) \mathcal{R}_{Nl}(L_p L_d),$$

where

$$\mathcal{R}_{Nl}(L_p L_d) = \int\limits_{r_0}^{\infty} r^2 dr\, e^{i\sigma_{L_p}(\nu_p) + i\sigma_{L_d}(\nu_d)} R_{Nl}(r)[F_{L_d} - \delta_{L_d} H_{L_d}][F_{L_p} - \beta_{L_p} H_{L_p}]$$

*) In the j-representation one has:

$$P_{I_f}(\ldots) = 4(I_i j_i K_i \Omega_n | I_f K_f)(I_i j_i' K_i \Omega_n | I_f K_f) C(j_i \Omega_n) C_{j_i j_i'}(\Omega_n) C_{j_i' j_i'}(\Omega_n)$$
$$= 4 \frac{2I_f+1}{2I_i+1} \beta_{j_i}(K_i \Omega_n) \beta_{j_i' j_i'}(K_i \Omega_n);$$

β_{j_i}s being the stripping reduced width (compare [1]).

in Tobocman's notation [9]. Now, if all the distortion effects are neglected, i. e. $\delta_\lambda = \beta_\lambda = \sigma_\lambda = 0$ then $F_L = j_L(kr)$, and we can quantize along $\vec{\varkappa}$ as in **2** with $S(l, m) = \delta_{0m}\sqrt{2l+1}\, e^{l}\mathcal{R}_{Nl}(\varkappa)$. The unimportant finite range factor $G(K)$, being independent of all the quantum numbers involved, can be omitted throughout. Then we have:

$$\Phi \sim \sum_\lambda Y_{2\lambda 0}(\theta_\gamma) \sum \{ll'jj'\}\, \xi^{l0l'0}_{jj'}(\Omega_n K_f I_f) \times$$

$$\times \sqrt{\frac{(2j+1)(2j'+1)}{2l+1}}\,(2\lambda l';\, 00\,|\,l0)\, F_{2\lambda}(LI_f'I_i)\, W(I_f I_i jj';\, 2\lambda I_i)\, W(ll'jj';\, 2\lambda\tfrac{1}{2}) \,. \tag{7}$$

It may be seen from (7) that, in this case, the omitted γ-rays are azimuthally symmetric about the axis of quantization.

Consider now, as an example, the reaction $^{25}\mathrm{Mg}\,(d,\,p)\,^{26*}\mathrm{Mg}\,(1.825\ \mathrm{MeV})\,(\gamma)\,^{26}\mathrm{Mg}$. The γ-transition from the first excited state 1.825 MeV to the ground state is a mixture of $M1$ and $E2$ — radiation according to May and Foster [10], provided one assumes $I_f = 2$ +**). For $L = 1$ with $\lambda = 0, 1$ we have an isotropic correlation function in view of $F_2(1;\,0,\,2) = 0$. On the other hand, for $L = 2$ ($E2$ transition), by using (6) we find:

$$\Phi \sim {}_0Y_{00}(\theta_\gamma) - 0.5976\, a_2 Y_{20}(\theta_\gamma) - 1.069\, a_4 Y_{40}(\theta_\gamma) \,, \tag{8}$$

where

$$a_2^\lambda = \sum \{ll'jj'\}\, \xi^{l0l'0}_{jj'} \sqrt{\frac{(2j+1)(2j'+1)}{2l+1}}\,(2\lambda' l;\, 00\,|\,l0) \times \tag{8a}$$

$$\times\, W(22jj';\, 2\lambda\tfrac{5}{2})\, W(ll'jj';\, 2\lambda\tfrac{1}{2}) \,,$$

$$\xi^{l0l'0}_{jj'} = P_2(ll'jj'\,|\,-\tfrac{1}{2}\tfrac{5}{2})\sqrt{2l+1}\sqrt{2l'+1}(-)^{(l-l')/2}\mathcal{R}_{2l}(\varkappa)\mathcal{R}_{2l'}(\varkappa) \,.$$

Explicitly:

$$a_0 = \frac{1}{\sqrt{5}}\left(\xi^{0000}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} - \frac{1}{5}\sqrt{\frac{2}{3}}\,\xi^{2020}_{\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}} + \frac{1}{5}\xi^{2020}_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}}\right),$$

$$a_2 = \frac{2}{25}\,(\xi^{2000}_{\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} + 2\xi^{2000}_{\frac{5}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}) + \frac{1}{5}\sqrt{\frac{2}{35}}\left(\frac{1}{2}\,\xi^{2020}_{\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}} - \frac{2}{7}\,\xi^{2020}_{\frac{3}{2}\frac{5}{2}\frac{3}{2}\frac{5}{2}} + \frac{2}{7}\,\xi^{2020}_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}}\right),$$

$$a_4 = \frac{1}{35\sqrt{5}}\left(\frac{2\sqrt{2}}{3}\,\xi^{2020}_{\frac{3}{2}\frac{5}{2}\frac{3}{2}\frac{5}{2}} + \sqrt{\frac{3}{14}}\,\xi^{2020}_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}}\right),$$

where

$$\xi^{0000}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} = \frac{10}{3}\,A_{00}^2|\mathcal{R}_{20}(\varkappa)|^2,\ \xi^{2020}_{\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}} = \frac{40}{21}\,(2A_{20}^2 + 3A_{2-1}^2 - 2\sqrt{6}A_{20}A_{2-1})|\mathcal{R}_{22}(\varkappa)|^2. \tag{9}$$

$$\xi^{2020}_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}} = \frac{5}{7}\,(3A_{20}^2 + 2A_{2-1}^2 + 2\sqrt{6}\,A_{20}A_{2-1})|\mathcal{R}_{22}(\varkappa)|^2,$$

$$\xi^{2000}_{\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} = -\frac{20}{3\sqrt{7}}\,A_{00}(\sqrt{2}\,A_{20} - \sqrt{3}\,A_{2-1})\mathcal{R}_{22}(\varkappa)\mathcal{R}_{20}(\varkappa),$$

*) However, if the deformed nucleus coupling is to apply, we have to assume a pure $E2$ — transition.

$$\xi^{2000}_{\frac{5}{2}\frac{1}{2}} = -5\sqrt{\frac{2}{21}}\left(\sqrt{3}A_{20} + \sqrt{2}A_{2-1}\right)A_{00}\mathcal{R}_{22}(\varkappa)\mathcal{R}_{20}(\varkappa),$$

$$\xi^{2020}_{\frac{3}{2}\frac{5}{2}} = \frac{10}{7}\sqrt{\frac{2}{3}}\left(\sqrt{6}A^2_{20} - \sqrt{6}A^2_{2-1}A_{20}A_{2-1}\right)|\mathcal{R}_{22}(\varkappa)|^2.$$

Unfortunately, no data on this angular distribution exist so far. The neutron wave function (i. e. the A-coefficients) for ^{26}Mg is not known, and, therefore, even no realistic theoretical numerical result can as yet be presented for Φ *).

2. Proton polarization

The polarization of protons from (d, p) reactions has been the subject of a number of papers (for references see e. g. [9]). It is interesting to see how the mixing of l's and j's affects the polarization of protons in the case of spheroidal nuclei. For the sake of simplicity no spin-orbit distortion on the outgoing proton wave is assumed. Let the quantization axis z be along the vector $\vec{k}_d$, and the x-axis in the reaction plane. The polarization pseudovector $\vec{P}$ is then easily calculated by using standard techniques. It is possible to show that the components of $\vec{P}$ in the reaction plane vanish, and the degree of polarization is:

$$(10)\qquad P \equiv P_y = \tfrac{1}{3}\Big\{\sum(2l+1)^{-1}\sum_{\mu_f=1/2}^{l+1/2}\sqrt{(l+\tfrac{1}{2})^2-\mu_f^2}\,Im[S(l\mu+\tfrac{1}{2})\times$$
$$\times S(l\mu_f\ \ \tfrac{1}{2})^*]\sum_{j=l-1/2}^{l+1/2}(\ \ )^{l+1/2-j}(2j+1)\ ^1P_l(lj|\Omega_n k_i)\Big|^2\times$$
$$\times\Big\{\sum_{lj}(2l+1)^{-1}P_l(lj|\Omega_n k_i)\sum_m|S(lm)|^2\Big\}^{-1},$$

where $S(l, m)$ is given by Eqs. (15) and (16). If no distortion effects are assumed (Butler approximation, $S(l, m) = \delta_{m0}S(1, 0)$ with Oz $\uparrow\uparrow \vec{k}$), there is no polarization as in the usual Butler (Born) theory. Eq. (10) becomes identical with Eq. (21) of Grant [11] provided one l and one j contribute.

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*) If, however, the 1.825 MeV level is regarded as a 2$^+$ member of the rotational band based on $K^\pi = 0^+$, a different expression for Φ is required: one has different $\xi^{lol'0}_{jj'}$ values.

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Kinematic Depolarization of μ-Mesons Emitted in π-Meson Decays

by

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The presence of an asymmetry in the angular distribution of electrons emitted in $\pi - \mu - e$ chain, predicted by the two-component variant of the neutrino field [1]-[3], has already been confirmed by several laboratories [4]-[10]. However, the differences between the values of the asymmetry parameter obtained by different laboratories are still much too considerable to warrant the drawing of quantitative conclusions, apart from that of violation of parity conservation [4]-[11]. Evidently, some part of these discrepancies can be explained by different experimental conditions. For example, different bubble chambers or nuclear emulsions, different external magnetic field, etc. may lead to different depolarization rates during the slowing down process of the μ-meson and consequently to different experimental values of the asymmetry parameter. Some aspects of the depolarizing influence of the stopping medium were discussed in [12]. However, cases are known, when two laboratories using exactly the same plates (exposed and developed in the same way) obtained quite different results. We shall therefore discuss another possible source of errors which is of purely kinematic origin and might have been overlooked or underestimated by some laboratories.

First, let us consider a monokinetic beam of π^+-mesons of momentum $\vec{P}$ decaying in flight into μ and ν. We assume that a μ^+-meson emitted by a π^+-meson decaying at rest (i. e. for $\vec{P} = 0$) is completely polarized in the direction of its momentum $\vec{p}_0$. Straightforward calculation leads to the following expression for the longitudinal component of the polarization vector of a μ^+-meson emitted by a π^+-meson of momentum $\vec{P}$

$$l = \frac{2(pE - \varepsilon P \cos\theta)}{m^2 - \mu^2}, \tag{1}$$

where θ is the angle between π^+-meson and μ^+-meson momenta: $\cos\theta = \frac{\vec{p}\vec{P}}{pP}$. The symbols $\mu, \vec{p}, \varepsilon$ and $m, \vec{P}, E$ denote the mass, momentum and energy of μ^+ and π^+, respectively. For a decay at rest $P = 0$, and we have

$$p_0 = \frac{m^2 - \mu^2}{2m}, \qquad \varepsilon_0 = \frac{m^2 + \mu^2}{2m}, \qquad E_0 = m. \tag{2}$$

Inserting (2) into (1) we find $l_0 = +1$, i. e. complete longitudinal polarization.

In general, the conservation equations for energy and momentum lead to the following expressions for μ-meson energy and momentum as functions of P and θ:

$$\varepsilon_{1,2} = \mu \frac{E'E \pm P\cos\theta \sqrt{P'^2 - P^2\sin^2\theta}}{m^2 + P^2\sin^2\theta}, \tag{3}$$

$$p_{1,2} = \mu \frac{E'P\cos\theta \pm E \sqrt{P'^2 - P^2\sin^2\theta}}{m^2 + P^2\sin^2\theta} \tag{4}$$

with

$$P' = \frac{m^2 - \mu^2}{2\mu}, \qquad E' = \frac{m^2 + \mu^2}{2\mu}. \tag{5}$$

Now we shall discuss separately two different cases:

Case A: If $P < P'$ or, in other words, if

$$E_{\text{kin}} = E - m < E'_{\text{kin}} = E' - m = 5.36 \text{ MeV} \tag{6}$$

we must reject the $(-)$ sign in (3) and (4). Inserting p_1 and ε_1 taken with the $(+)$ sign we find for this region of π-meson energies

$$l_1 = + \sqrt{1 - P^2/P'^2\sin^2\theta}. \tag{7}$$

Thus, we see that, for a fixed P, the longitudinal component of the polarization vector of μ^+ depends on the angle θ. It has a minimum for $\theta = \frac{\pi}{2}$ (see Fig. 1).

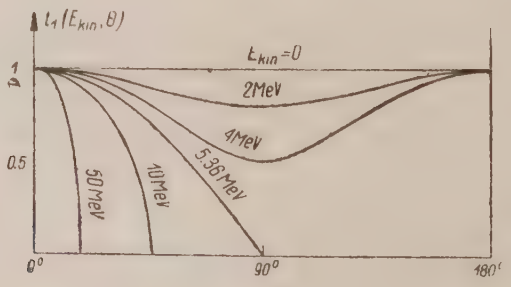

Fig. 1

Case B: If $P > P'$, i. e. if

(8) $$E_{\text{kin}} > 5.36 \text{ MeV},$$

the angles θ are restricted to the interval

(9) $$0 \leqslant \theta \leqslant \theta_k = \arcsin \frac{P'}{P} \leqslant 90^\circ ,$$

but for any P and θ from this interval we have now two values of ε and p given by (3) and (4). Inserting these values into (1) we find correspondingly the following two values of l

(10) $$l_{1,2} = \pm \sqrt{1 - P^2/P'^2 \sin^2\theta} .$$

Thus, we see that, in this second region of π-meson energies, the beam of μ-meson emitted at an angle θ consists actually of two beams of different energies, ε_1, ε_2, and opposite polarizations $l_2 = -l_1$. It should be noted that, for $\theta \to \theta_k$, the difference $\varepsilon_1 - \varepsilon_2 \to 0$. A few plots of $l_1(P, \theta)$ for some fixed values of $E_{\text{kin}} = \sqrt{m^2 + P^2} - m$ are shown in Fig. 1.

The normalized probability of finding the μ-meson with momentum p_i given by (4) emitted at an angle θ has the form

(11) $$W_i(P, \theta) = \frac{p_i^2}{4\pi p_0^2 \sqrt{1 - P^2/P'^2 \sin^2\theta}} .$$

By means of formulae (3)-(11) we can now calculate all the necessary averages of l according to the kind of experiment.

As an example, let us calculate the mean value of l averaged over all angles for a fixed P. For the sake of simplicity we shall restrict ourselves to the case $P < P'$, i. e. to $E_{\text{kin}} < 5.36$ MeV. For this range of π-meson energies we find

(12) $$\tilde{l}(P) = 2\pi \int_0^\pi W_1(P, \theta) l_1(P, \theta) \sin\theta \, d\theta = \frac{1}{V'^2} - \frac{1 - V'^2}{2V V'^2} \ln \frac{1+V}{1-V} =$$

$$= 1 - \frac{1 - V'^2}{3} \frac{V^2}{V'^2} \left(1 + \frac{3}{5} V^2 + \frac{3}{7} V^4 + \ldots\right),$$

where

(13) $$V = \frac{P}{E}, \qquad V' = \frac{P'}{E'} = 0.27 .$$

For $P = P'$, i. e. for $E_{\text{kin}} = 5.36$ MeV, we find that $\tilde{l}(P') = 0.677$. We see that $\tilde{l}(E_{\text{kin}})$ decreases quite rapidly with increasing π-meson energy (Fig. 2). Neglecting terms of higher order in V^2, we can reduce (12) to the following much simpler form

(14) $$\tilde{l}(E_{\text{kin}}) = 1 - 0.061 E_{\text{kin}} .$$

In practice, we always have a continuous spectrum of π-meson momenta in the beam which can be characterized by the probability $R(P)dP$. If we take all possible decays in such a beam without discrimination, we must average (12) over all π-meson momenta. The final mean value of l will then be given by the formula

$$\tilde{l} = \frac{\int_0^\infty R(P)\tilde{l}(P)dP}{\int_0^\infty R(P)dP}. \tag{15}$$

In order to avoid the difficulties connected with decays in flight, many laboratories selected only cases that could be ascribed to decays

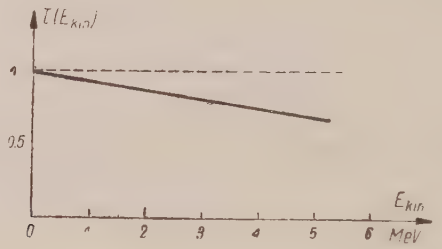

Fig. 2

at rest. Since measurements of π^+-meson momentum are often not accurate enough, the length of the μ^+-meson track, i. e. μ-meson energy, is used as a check distinguishing a decay at rest from a decay in flight. However, it can easily be seen that for any $P < P' \frac{2E'}{m}$ a value θ'' such that either

$$\varepsilon_1(P, \theta'') = \varepsilon_0 \quad \text{or} \quad \varepsilon_2 = (P, \theta'') = \varepsilon_0 \tag{16}$$

can always be found.

In this case, the mean value of l averaged over all angles and momenta will be given by

$$\tilde{l} = \frac{\sum_{i=1}^{2} \int\int R(P)W_i(P, \theta)l_i(P, \theta)G(\varepsilon_i - \varepsilon_0)\sin\theta dP d\theta}{\sum_{i=1}^{2} \int\int R(P)W_i(P, \theta)G(\varepsilon_i - \varepsilon_0)\sin\theta dP d\theta}, \tag{17}$$

where function G characterizes the accuracy of μ^+-meson track measurements. Thus, we see that, even in the present case, one may have some contribution from decays in flight, which will result in a lowering of the experimental value of the asymmetry parameter. However, it follows from (3), (4) and (16) that the angles θ'' lie in the interval

$0 < \theta'' < 90°$. Therefore, if the length of the track is measured with sufficient accuracy, it is possible to get rid of the errors connected with kinematic depolarization by selecting only such cases, where $90° < \theta < 180°$.

Summarizing, we can say that kinematic effects may lead to a fairly substantial lowering of the mean longitudinal polarization of μ-mesons. Mean value of l depends, in general, on a few empirical probability functions $R(P)$, $G(\varepsilon_i - \varepsilon_0)$, etc., which may have quite different forms for different experiments. Kinematic effects can therefore be partly responsible for some part of the confusion that still exists in π-μ-e experiments.

This work was accomplished during my visit to the United Institute for Nuclear Research at Dubna.

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BULLETIN DE L'ACADÉMIE
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The Longitudinal Polarization of Electrons in Once Forbidden β Transitions

by

A. DELOFF

Presented by L. INFELD on November 18, 1957

According to the two-component neutrino theory [1], [2], longitudinal polarization of β electrons in allowed transitions is equal to $\pm\frac{v}{c}$ for all pure interactions and gives a restricted possibility of distinguishing between various pure interactions. The aim of this paper is to investigate polarization in the case of once forbidden β^- transitions.

We define the polarization

$$P_i = \frac{I_i^+ - I_i^-}{I_i^+ + I_i^-}, \tag{1}$$

where $I_i^+(W)$ and $I_i^-(W)$ are the probabilities of finding the electron with energy W and with spin projection on the momentum direction equal to $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. We confine our attention solely to the pure interactions, i being as usual S, P, A, T, V.

For the probability $I_i^{\pm}$ we have

$$I_i^{\lambda} = C|H_i^{\lambda}|^2, \tag{2}$$

where $\lambda = \pm 1$ and C are multiplicative energy dependent factors. If the two-component neutrino theory is used, the interaction Hamiltonian H_i has the following form given in [1], [2]

$$H_i = (V^* O_i U) \quad (\psi^* O_i[1+\gamma_5]\varphi). \tag{3}$$

The Coulomb effect is treated approximatively, according to a method given by Cutkosky [3], where the electron wave function is represented by:

$$\Psi(\vec{r}, \vec{p}, \sigma) \cong \tag{4}$$

$$\cong (1 + i\vec{p}\vec{r} + i\xi\vec{\alpha}\vec{r} - \tfrac{1}{2}(\vec{p}\vec{r})^2 - \tfrac{1}{2}\xi(\vec{\alpha}\vec{r})(\vec{p}\vec{r}) + \ldots)u(\vec{p}, \sigma), \quad \hbar = c = m = 1,$$

where $\vec{p}$ is the asymptotic momentum of the electron, $u(\vec{p}, \sigma)$ is the plane wave spinor, $\vec{\alpha}$ – the usual Dirac operator and $\xi = \frac{\alpha Z}{2R} \sim Z$; where Z is the final nucleus charge, $\alpha = \frac{1}{137}$ and R are the nuclear radii. Formula (4) is valid for $\alpha Z \ll 1$, $pR \ll 1$ and $\frac{\alpha Z}{pR} \sim 1$ and is very useful because it makes possible the use of projection operators and trace techniques. We derive I_i' by inserting (3) into (2) and choosing the corresponding spin states for an electron with the help of the spin projection operator $\frac{1}{2}\left(1 + \lambda \frac{\vec{\alpha}\vec{p}}{p}\right)$.

After inserting I_i' into (1) we obtain the polarization as function of electron energy W. The expressions for $P_i(W)$, for all five types of interaction, are given in (5). For $\lambda = 0$ the expression I_i^0 becomes the total probability of a β^- transition as given, for example, by Konopinski and Uhlenbeck [4].

A superficial inspection of (5) seems to indicate that there is a deviation from the $v\,c$ law. However, the transition energies W_0 and W do not exceed 10 in our units, whereas the dominant terms are always those proportional to ξ^2 which, for $Z \gtrsim 30$, are of the order 10^3. For this reason all the Δ terms can be neglected in comparison with the remaining terms and it becomes clear that the method proposed by Cutkosky does not lead to any considerable deviations of the polarization curves from the $\frac{v}{c}$ law.

The error introduced by using method [3] seems to be smaller in the case of polarization than in the case of transition probability. For example, a Z dependent multiplicative factor occurring in transition probability does not occur in polarization, even in the exact calculus.

Recent experiments on longitudinal polarization [5] made for Au^{198} gave for energy 168 keV the suprisingly low polarization value $P = -0.16 \pm 0.03$, whereas $\frac{v}{c}$ for this energy is 0.66. It seems impossible to explain so great a discrepancy by pure interaction.

$$P_S = P_P = -\frac{p}{W}\frac{A}{A + \Delta},$$

$$A = \tfrac{8}{3}W^2 + 4\left(\xi - \tfrac{2}{3}W_0\right)W + \left(2\xi W_0 + W_0^2 - 1 + 3\xi^2\right)$$

$$\Delta = \frac{2}{3}\frac{W_0}{W} \quad \frac{2}{3} \quad \frac{2}{3}\frac{\xi}{W};$$

$$P_T \qquad \frac{p}{W}.$$

$$\tfrac{1}{12}|B_{ij}|^2 A + \tfrac{1}{6}\left|\int \vec{\sigma} \times \vec{r}\right|^2 r^2 B + \left|f\dot{\alpha}\right|^2 + \tfrac{1}{9}\left|\int \vec{\sigma} \cdot \vec{r}\right|^2 C - \left[\left(\int \vec{\sigma} \times \vec{r}\right)\left(\int \vec{\alpha}\right)^* + c.c.\right] D$$

$$\tfrac{1}{12}|B_{ij}|^2 A + \tfrac{1}{6}\left|\int \vec{\sigma} \times \vec{r}\right|^2 (B + \Delta_1) + \left|\int \vec{\alpha}\right|^2 + \tfrac{1}{9}\left|\int \vec{\sigma} \cdot \vec{r}\right|^2 (C + \Delta_2) - \left[\left(\int \vec{\sigma} \times \vec{r}\right)\left(\int \vec{\alpha}\right)^* + c.c.\right](D +$$

$$A = 2W^2 - 2W_0W + (W_0^2 - 1);$$

$$B = \tfrac{2}{3}W^2 - \tfrac{2}{3}W_0W + (W_0^2 - 1 + 6\xi^2 + 4\xi W_0);$$

$$\Lambda_1 = -4\left(\frac{\xi}{W} + \frac{W_0}{3W} - \frac{1}{3}\right)$$

$$C = 4W^2 + 4(3\xi - W_0)W + (W_0^2 - 1 + 9\xi^2 - 6\xi W_0);$$

$$\Lambda_2 = -6\left(\frac{\xi}{W} - \frac{W_0}{3W} + \frac{1}{3}\right)$$

$$D = \frac{W_0}{3} + \xi; \qquad \Lambda_3 = -\frac{1}{3W}$$

$$(5)\quad P_e = +\frac{p}{W}\frac{\int \vec{r}^2 A + \int \vec{a}^2 - i\left\{(\int \vec{r})^*(\int \vec{a}) - c.c.\right\} B}{\int \vec{r}^2(A + \Lambda_1) + \int \vec{a}^2 + i\left\{(\int \vec{r})^*(\int \vec{a}) - c.c.\right\}(B + \Lambda_2)}$$

$$A = \tfrac{2}{3}W^2 - \tfrac{2}{3}W_0W + (\tfrac{2}{3}\xi W_0 - \tfrac{1}{3} + \tfrac{1}{3}W_0 + \xi^2); \quad \Lambda_1 = -\frac{2}{3}\left(\frac{W_0}{3W} - \frac{1}{3} + \frac{\xi}{W}\right)$$

$$B = \tfrac{1}{3}W_0 + \xi; \qquad \Lambda_2 = -\frac{1}{3W}$$

$$P_A = +\frac{p}{W}\times$$

$$\frac{\tfrac{1}{6}|B_{ij}|^2 A - \tfrac{1}{6}|\int \vec{\sigma}\times\vec{r}|^2 B + |\int \vec{\sigma}\vec{r}|^2 C + |\int \gamma_5|^2 + i\left\{(\int \vec{\sigma}\vec{r})^*(\int \gamma_5) - c.c.\right\} D}{\tfrac{1}{6}|B_{ij}|^2 A + \tfrac{1}{6}|\int \vec{\sigma}\times\vec{r}|^2(B + \Lambda_1) + |\int \vec{\sigma}\vec{r}|^2(C + A) + |\int \gamma_5|^2 + i\left\{(\int \vec{\sigma}\vec{r})^*(\int \gamma_5) - c.c.\right\}(D + \Lambda_3)}$$

$$A = 2W^2 - 2W_0W + (W_0^2 - 1);$$

$$B = \tfrac{2}{3}W^2 + 2(4\xi - 2W_0)W + (6\xi^2 - 4\xi W + W_0^2 - 1);$$

$$\Lambda_1 = -4\left(\frac{\xi}{W} - \frac{W_0}{3W} + \frac{1}{3}\right)$$

$$C = \tfrac{16}{9}W^2 - \tfrac{16}{9}W_0W + (W_0^2 - 1 - \tfrac{2}{3}\xi W_0 + \xi^2); \quad \Lambda_2 = -\frac{2}{3}\left(\frac{\xi}{W} + \frac{W_0}{3W} - \frac{1}{3}\right)$$

$$D = \tfrac{1}{3}W_0 + \xi; \qquad \Lambda_3 = -\frac{1}{3W}.$$

The same notation as in [4] is adopted (energy in mc^2 units).

These expressions were found independently and they are in agreement with those obtained recently in [6].

I am much indebted to Dr. J. Werle for suggesting the problem and for helpful discussions.

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BULLETIN DE L'ACADÉMIE
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Investigation of Ferroelectrics by Temperature Measurements

by

A. PILAWSKI

Presented by A. SOŁTAN on October 18, 1957

The rise in temperature ΔT produced within a barium titanate ceramic sample by an AC field at a frequency of 50 cycles per second has been measured. Experiments were made at various temperatures T of the sample. In some cases a biasing DC field was superimposed on the sample.

On the assumption that the rise in temperature may be considered to correspond to the losses, the author was able to investigate the variation of the latter with the temperature and the strength of the biasing field.

Measurements

The $BaTiO_3$ ceramic sample *) was placed in a test-glass where it remained suspended on thin conductors soldered to its silver electrodes. One junction of the thermocouple was soldered on one of the electrodes, the other being attached to a thermometer container also placed within the test-glass. The latter, together with its contents, was immersed in the oil of an ultrathermostat.

The difference in temperature ΔT appearing between the sample and its environment upon application of an AC voltage was obtained from a galvanometer with an accuracy of 0.2°C. The temperature of the environment was indicated by the thermometer with an accuracy of 1°C. The reading of the difference ΔT between the temperature of the sample and that of its environment could be made when the temperature was established, i. e. after about 20-30 minutes. This difference of temperature was measured at various environmental temperatures obtained by heating the thermostatic liquid.

*) The $BaTiO_3$ ceramic sample was prepared by Z. Pająk at the Institute of Experimental Physics, A. Mickiewicz University, Poznań, by a method which will be published in Acta Physica Polonica.

Similarly, measurements were made with the sample subjected to a biasing DC field.

The scheme for applying to the sample a DC or an AC voltage, or both simultaneously, is shown in Fig. 1.

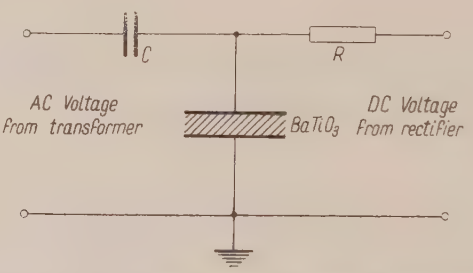

Fig. 1. Scheme for applying an AC field and a biasing field to the sample. $C = 1\mu F$, $R = 100\ M\Omega$.

Results

The results of measurements are represented by the curves in Fig. 2. Curve 1 shows the difference in temperature ΔT between the sample and its environment versus temperature T of the sample, the AC field being 2650 V/cm. and the frequency 50 cycles per second. Curves 2, 3, 4 correspond to a superimposed biasing DC field of 1350, 1350 and 5300 V/cm. respectively.

Conclusions

The rise in temperature of the $BaTiO_3$ sample upon application of an AC electric field is caused by losses resulting from hysteresis and conductivity of the sample.

The power loss of the sample is given by Newton's law:

$$P = \alpha \cdot s \cdot \Delta T\,, \tag{1}$$

where α denotes the coefficient of cooling, s the surface area and ΔT — the temperature difference.

At the same time, the input power of the sample is:

$$P = U^2 \omega C \operatorname{tg}\delta\,, \tag{2}$$

where U denotes the voltage applied, ω the angular frequency, C the capacity and $\operatorname{tg}\delta$ — the loss tangent.

In steady state, the power loss of the sample is equal to that of its input power. Thus, we have:

$$\Delta T = k \cdot \varepsilon \cdot \operatorname{tg}\delta\,,$$

with k denoting a coefficient, and ε — the dielectric constant.

The difference in temperature ΔT between the sample investigated and its environment is a linear function of the product $\varepsilon \mathrm{tg}\delta$, inasmuch as the coefficient k may by considered to be independent of temperature.

According to (2), the product $\varepsilon \mathrm{tg}\delta$ accounts for the losses within the sample, and thus the temperature difference ΔT should also, to some

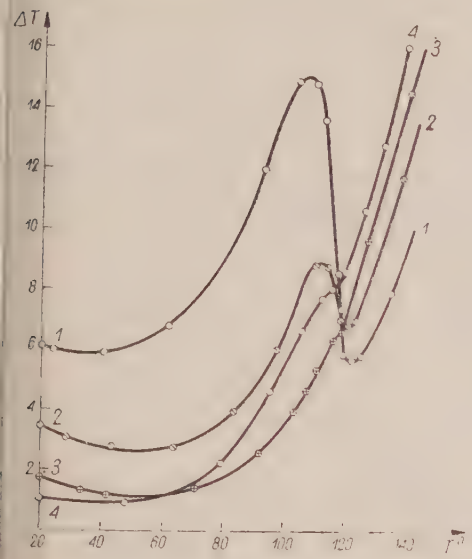

Fig. 2

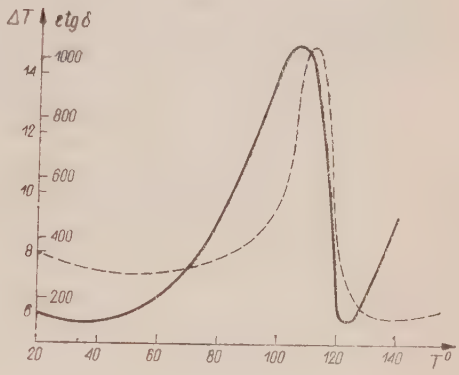

Fig. 3

Fig. 2. Difference ΔT between temperature of sample and that of environment versus sample temperature T. AC field 2650 V/cm.
Biasing DC field curve 1—0 V/cm.
„ 2—1350 „
„ 3—4350 „
„ 4—5300 „

Fig. 3. ——— curve 1 of Fig. 2
– – – $\varepsilon \mathrm{tg}\delta$ versus temperature according to Mesnard and Eyraud [1].

extent, account for them. Following this point of view the author concludes that:

1) In a strong AC field as high as 2650 V/cm., frequency 50 cycles per second (Fig. 2, curve 1), losses grow as the temperature rises, beginning from a temperature of approximately 40°C., and attain a maximum at about 110°C. Upon further rise of temperature, the losses fall suddenly attaining a minimum about the Curie point at 120°C. On passing the Curie point the losses grow rapidly.

The above conclusions were compared with the results of G. Mesnard and L. Eyraud [1] obtained under different conditions. The dielectric constant ε and the loss tangent $\mathrm{tg}\delta$ were measured by the bridge method, a weak AC field of the order of 10^1 V/cm. being applied at a frequency of 50 cycles per second. In order to compare results, curve 1 from Fig. 2 (continuous line) and $\varepsilon \mathrm{tg}\delta$ versus temperature (dashes) obtained from the data in [1] are shown in Fig. 3. The results agree at least

qualitatively; some divergences are to be explained by differences in field strengths and in sample quality.

2) If a biasing DC field is superimposed on the AC field applied to the sample, the losses fall for temperatures below the Curie point. A number of authors [1]-[3] obtained similar results. The rise in losses observed on applying strong DC fields above 80°C. may be explained as an effect of the appearance of conductivity; this rise was also observed by Mesnard and Eyraud [1].

Moreover, it could be observed that a sample heated with an AC field begins to cool as soon as a DC field is applied (Fig. 4).

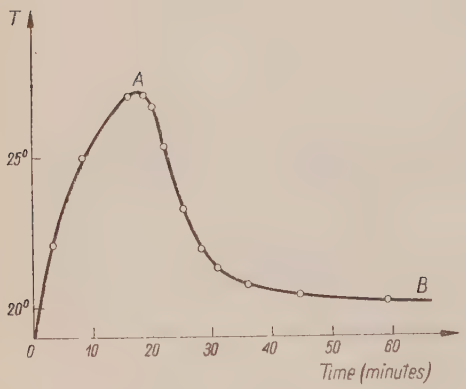

Fig. 4

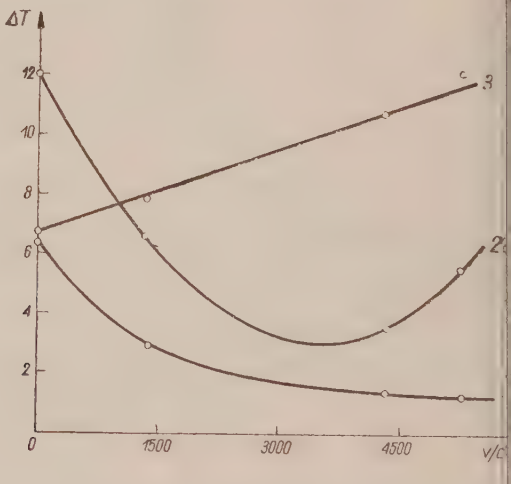

Fig. 5

Fig. 4. Temperature of $BaTiO_3$ sample versus time. Section 0 to A: in AC field 3000 V/cm section A to B: in AC field of 3000 V/cm. with superimposed DC field of 3000 V/cm

Fig. 5. Temperature difference ΔT between $BaTiO_3$ sample and environment versus DC biasing field strength at constant AC field strength of 2650 V/cm. and at various temperatures of sample: curve 1—60° C, curve 2—100° C, curve 3—130° C.

3) The dependence of ΔT on the strength of the DC field at constant AC field and constant temperature of the sample, obtained from the diagrams in Fig. 2, is shown in Fig. 5.

It may be deduced from the curves that:

a) at temperatures below c. 60°C., losses fall as DC field strength rises;

b) within a range of temperatures around 100°C. losses decrease as stronger fields are applied, and attain a minimum value for a critical field, where upon stronger fields produce a rise in the losses;

c) above the Curie point, losses grow almost linearly with the biasing field.

The author wishes to express his indebtedness to Professor A. Pic-kara, for his valuable suggestions and for providing $BaTiO_3$ samples from the Institute.

Summary

1. The difference ΔT between the temperature of the $BaTiO_3$ ceramic sample and that of its environment, caused by the effect of the AC field of a frequency of 50 cycles per second, depends on temperature and reaches a maximum at about 110°C., and a minimum at the 120°C. Curie point.

2. A biasing DC field reduces the temperature difference at temperatures below the Curie point, whilst raising it at higher temperatures.

3. The effect of a biasing DC field on the difference of temperatures ΔT varies in accordance with the temperature range:

— below 60°C. approximately, ΔT falls with stronger DC field;

— within a range of temperatures around 100°C., ΔT reaches a minimum at a certain field value;

— above 120°C., ΔT rises with the temperature.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФЗИИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 1

Р. М. ФРИДБЕРГ, 4-КВАНТОРНОЕ ДОПОЛНЕНИЕ: НЕРАВНОМЕРНО ЧАСТИЧНО РЕКУРСИВНЫЙ ОПЕРАТОР БАНАХА-МАЗУРА . . . . . стр. 1—5

Пусть Θ_u — универсальная частично рекурсивная функция, аргументами которой являются конечные последовательности $\langle q_0, q_1, \ldots, q_n\rangle$, а значениями — натуральные числа и пусть $\Theta_u(f) = \lim_{n\to\infty} \Theta_u(\langle f(0), f(1), \ldots, f(n)\rangle)$ для u таких, что Θ_u определено для всех конечных последовательностей и существует правая часть. Пусть $A(u)$ является следующим предикатом: $\Theta_u^*(f)$ определено для всех рекурсивных f и для этих f совпадает со значениями некоторого равномерно частично рекурсивного оператора.

Автор доказывает, что предикат $A(u)$ является полным (в смысле Поста) для класса предикатов вида $(Ex)(y)(Ez)(w)R(u,x,y,z,w)$, где R — рекурсивно.

Как следствие, автор получает теорему о том, что существует оператор Банаха-Мазура, который не совпадает ни с каким оператором, образованным из равномерно частично рекурсивного оператора путем ограничения аргументов к рекурсивным функциям.

С. МРУВКА, О МОЩНОСТИ КОМПАКТНЫХ ПРОСТРАНСТВ И О ПЕРВОЙ АКСИОМЕ СЧЕТНОСТИ . . . . . стр. 7—6

В работе изучается связь между мощностью компактного пространства и количеством точек счетного характера (т. е. точек, у которых имеется счетная база окрестностей). Доказана следующая теорема:

Пусть X — компактное пространство. Если X не удовлетворяет первой аксиоме счетности во всякой точке, тогда $\overline{X} \geqslant 2^{\aleph_1}$.

Следствиями этой теоремы являются следующие утверждения:

1. *Всякое не пустое замкнутое множество типа G_σ, лежащее в компактном пространстве, имеет мощность $\geqslant 2^{\aleph_1}$; в частности, всякое не пустое замкнутое множество типа G_σ, лежащее в βX X, где βX обозначает максимальное компактное расширение пространства X, имеет мощность $\geqslant 2^{\aleph_1}$.*

2. *Если X — несчетное компактное пространство мощности $< 2^{\aleph_1}$, то X содержит несчетное количество точек счетного характера.*

А. ПЛИСЬ, О ТОЧКАХ КАСАНИЯ ДВУХ ИНТЕГРАЛОВ УРАВНЕНИЯ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ ПЕРВОГО ПОРЯДКА . . . стр. 11—14

Известно, что уравнение

$$(I) \qquad z_x = f(x, y_1, \ldots, y_n, z, z_{y_1}, \ldots, z_{y_n})$$

с правой стороной класса C^2, а также решения уравнения (I) класса C^2 обладают следующим свойством:

если два интеграла соприкасаются в некоторой точке, то они являются касательными вдоль соответственной характеристики.

В работе приведены некоторые обобщения этой теоремы и дан пример, указывающий на то обстоятельство, что это свойство не обязательно должно иметь место для уравнения (I) с правой стороной класса C^1.

А. БЕЛЕЦКИЙ и Я. КИСЫНЬСКИЙ, ЗАМЕТКА В СВЯЗИ С ДВУМЯ СООБЩЕНИЯМИ С. ШМЫДТ . . . . . стр. 15—17

Предположим, что для $i = 1,2$ символы E_i и $\delta_i(\xi_i, \eta_i)$ обозначают полное метрическое пространство и расстояние его элементов, а $\widetilde{\xi}_i = \Phi_i(\xi_1, \xi_2)$ — преобразование декартова произведения $E^* = E_1 \times E_2$ в пространство E_i, удовлетворяющее условию

$$\delta_i(\widetilde{\xi}_i, \widetilde{\eta}_i) \leqslant a_{i1}\,\delta_1(\xi_1, \eta_1) + a_{i2}\,\delta_2(\xi_2, \eta_2).$$

Если положительные постоянные a_{ij} удовлетворяют условиям a_{11} 1. $a_{22} < 1$ и $a_{12}a_{21} < (1 \quad a_{11})(1 \quad a_{22})$, то существует одна и только одна неподвижная точка преобразования $(\widetilde{\xi}_1, \widetilde{\xi}_2) = (\Phi_1(\xi_1, \xi_2), \Phi_2(\xi_1 \xi_2))$ в пространство E^*. Это весьма легко доказать, вводя соответственную метрику в пространстве E^* и применяя теорему о неподвижной точке Банаха. Лемма применяемая С. Шмыдт в двух работах [2], [3] является частным случаем этого утверждения. Одновременно, получаем сокращение доказательства, приведенного С. Шмыдт.

М. АЛЬТМАН, О ПРИБЛИЖЕННОМ РЕШЕНИИ НЕЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ В ПРОСТРАНСТВАХ БАНАХА . стр. 19—24

Настоящая заметка содержит дальнейшее исследование предложенного автором применения метода Ньютона приближенного решения функциональных уравнений $F(x) = 0$, где F — нелинейный функционал в пространстве Банаха. Основную роль играет здесь принцип мажорант, развитый Л. В. Канторовичем.

Применение принципа мажорант к исследованию метода Ньютона-Канторовича сводится Канторовичем к применению этого принципа к некоторому процессу последовательных приближений.

В данном случае это невозможно и, следовательно, необходимо ввести некоторое видоизменение принципа Канторовича. Применение принципа мажо-

рант позволяет нам получить новые условия сходимости обобщения метода Ньютона, предложенного автором, которые одновременно обеспечивают существование решения. Кроме того, получаем обобщение некоторых ранее приведенных результатов и упрощение теории.

С. ШМЫДТ, О СУЩЕСТВОВАНИИ РЕШЕНИЙ НЕКОТОРЫХ КРАЕВЫХ ЗАДАЧ КАСАЮЩИХСЯ СИСТЕМЫ ГИПЕРБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ . . . . . стр. 31—36

Автор приводит одно общее доказательство на существование решения для двух поставленных ею в работе [3] краевых задач, касающихся системы n-уравнений

$$u^{(i)}_{xy}(x,y)=f^{(i)}(x,y,u^{(n)},\ldots,u^{(1)},u^{(1)}_x,\ldots,u^{(n)}_x,u^{(1)}_y,\ldots,u^{(n)}_y) \quad (i=1,\ldots,n).$$

Кроме того, автором даются обобщения некоторых результатов, полученных ею в предыдущих работах [3]-[5].

И. САВИЦКИЙ, ВЫТАЛКИВАНИЕ ДЕЙТЕРОНА НА СФЕРОИДАЛЬНОМ НУКЛОНЕ. I. . . . . стр. 37—44

Предложено пояснение эффекта выступания двух и более значений l_n в угловых распределениях при выталкивании дейтерона — с точки зрения модели деформированного ядра. Волновые функции выхваченного нейтрона представлены при использовании волновых функций С. Г. Нильссона. Продискутированы общие свойства оперативных сечений и правило выборки. Получено качественное согласие с опытом для реакции (d,p) на Mg^{25} и Al^{27}.

И. САВИЦКИЙ, ВЫТАЛКИВАНИЕ ДЕЙТЕРОНА НА СФЕРОИДАЛЬНОМ НУКЛОНЕ. II. . . . . стр. 45—49

Даются общие формулы для угловых распределений с радиацией, сопутствующей реакции (d,p) на деформированных ядрах с $A\approx 25$-18.

Применены волновые функции сфероидальной модели, данные С. Г. Нильссоном для представления выхваченного в процессе (d,p) нейтрона. Приводятся формулы для поляризации выходящих протонов.

Я. ВЕРЛЕ, КИНЕМАТИЧЕСКАЯ ДЕПОЛЯРИЗАЦИЯ μ-МЕЗОНОВ, ЭМИТТИРОВАННЫХ В ПРОЦЕССЕ РАСПАДА π-МЕЗОНОВ . . . . стр. 51—55

В работе продискутирована степень продольной поляризации μ-мезонов, эмиттированных π-мезонами обладающими конечным моментом. Доказано, что в распадах, совершающихся во время пролета, средняя степень продольной поляризации падает очень быстро по мере уменьшения скорости π-мезонов. Этот чисто кинематический эффект можно, таким образом, считать причиной не согласующихся экспериментальных результатов, полученных в исследованиях распадов.

А. ДЕЛЛОФФ, ПРОДОЛЬНАЯ ПОЛЯРИЗАЦИЯ ЭЛЕКТРОНОВ В ОДНОКРАТНО ЗАПРЕЩЕННЫХ β-ПЕРЕХОДАХ . . . . . стр. 57—59

Исчислена степень поляризации электронов β в однократно запрещенных переходах. Применяя метод Катковского автор доказал, что в данном им приближении степень поляризации для чистых взаимодействий равна $\pm v/c$, аналогично как в случае разрешенных переходов.

А. ПИЛЯВСКИЙ, ИССЛЕДОВАНИЕ ФЕРРОЭЛЕКТРИКОВ ПРИ ПОМОЩИ ИЗМЕРЕНИЙ ТЕМПЕРАТУРЫ . . . . . стр. 61—65

1. Разница ΔT между температурой керамического образца $BaTiO_3$ и температурой его среды, вызванная воздействием поля переменного с частотой 50 *гц* находится в зависимости от температуры; она достигает своего максимума при 110°C, а минимума в точке Кюри 120°C.

2. Постоянное поле, действующее вместе с переменным, редуцирует эту разницу при температурах ниже точки Кюри, тогда как выше точки Кюри эта разница возрастает.

3. Влияние смещающего поля на разницу температур ΔT изменяется в зависимости от пределов температуры:

— ниже ок. 60°C ΔT уменьшается при более сильных напряжениях постоянных полей;

— в пределе температуры ок. 100°C ΔT достигает своего минимума при некотором значении поля;

— выше 120°C ΔT возрастает с ростом температуры.

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